On the Teukolsky Equation.

G. Marcilhacy

Laboratoire de Physique Théorique, E.R.A. 533
Institut Henri Poincaré - 11, rue P. et M. Curie, 75231 Paris Cedex 05

(ricevuto il 5 Aprile 1983)

PACS. 04.20. - General relativity.

One knows that a partial differential wave equation describing the dynamical gravitational, electromagnetic, scalar and neutrino field perturbations $\psi$ of a Kerr rotating black hole, has been obtained by Teukolsky (1). In Boyer-Lindquist co-ordinates, for example (2), this equation is a separable equation. The corresponding radial part of this equation is, with

$$
\psi = \exp[-i\omega t]R(r) \exp[-im\phi]S_\ell^m(\theta),
$$

(1) $\left(\frac{d}{dr} + \frac{1}{r^2} \frac{d}{dr} \left( r^{l+1} \frac{d}{dr} \right) + \left\{ \frac{K^2 - 2is(r - M)K}{\Delta} + 4i\omega r - \lambda \right\} \right) R = 0,$$

where

$$
K = (r^2 + a^2)\omega - am,
$$

$$
\Delta = r^2 - 2Mr + a^2,
$$

$$
\lambda = a^2\omega^2 - 2am\omega + A(a\omega);
$$

$A(a\omega)$ is the separation constant and $a$ the angular momentum, whereas the angular equation for the functions $S_\ell^m(\theta)$ is

$$
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) +
\left[ \frac{rr^2 \cos^2 \theta - m^2}{\sin^2 \theta} - 2a\cos \theta \frac{2ms \cos \theta}{\sin^2 \theta} - s^2 \cot^2 \theta + s + A \right] S = 0.
$$

The $S^{m}_{l}(\theta)$ are the so-called, spin-weighted angular spheroidal wave functions and when $s = 0$, $S^{m}_{l}(\theta)$ are the usual spheroidal functions.

Until now it would seem that these two equations have been investigated separately.

The present paper deals with the angular and radial parts of the Teukolsky equation, showing that—in fact—these two differential equations are particular forms of a single linear differential equation. These two equations can be derived by coalescence of singular points, from a differential which has eight elementary singularities (3).

This equation which has 8 elementary singularities situated at the points, $a_1, a_2, \ldots, \infty$, is

$$
\frac{d^2W}{dz^2} + \left[ \sum_{r=1}^{7} \frac{3}{z-a_r} \right] \frac{dW}{dz} + \left\{ \sum_{r=1}^{7} \frac{a_r(a_r+\frac{1}{2})}{(z-a_r)^2} + \frac{A_0 + A_1 z + \ldots + A_8 z^8}{\prod_{r=1}^{7} (z-a_r)} \right\} W = 0
$$

with

$$A_8 = \left[ \sum_{r=1}^{7} a_r \right]^2 - \sum_{r=1}^{7} \frac{a_r^2}{z-a_r} - 3 \sum_{r=1}^{7} a_r + \frac{3}{2}.$$

By coalescence of singularities, an equation with two nonelementary regular singularities and one regular singularity of the second species at infinity, can be carried out of eq. (3). This equation assumes the form

$$
\frac{d^2W}{dz^2} + \frac{C_1 z + C_0}{(z-z_1)(z-z_2)} \frac{dW}{dz} + \frac{B_0 + B_1 z + \ldots + B_4 z^4}{(z-z_1)^2(z-z_2)^2} W = 0.
$$

This linear equation is characterized by the formula $[0, 2, 1, 1]$ (4). The 9 arbitrary constants of this eq. (4) can be reduced to 5 through elementary transformations; thus eq. (4) becomes

$$
\frac{d^2W}{dz^2} + \left[ \frac{\lambda}{z} + \frac{\mu}{z-1} \right] \frac{dW}{dz} + \frac{C_0 + C_1 z + C_2 z^2}{4z(z-1)} W = 0.
$$

The 5 arbitrary constant $\lambda, \mu, C_0, C_1, C_2$, are irreducible constants, $z = (0, 1)$ are nonelementary, regular singular points and $z, \infty$ one irregular singularity of the second species.

Persidès (3) showed that the radial equation (1) was of the type (5). Fackerell and Crossman (5) seeking a representation of the spin-weighted angular spheroidal functions with the Jacobi polynomials, showed that it can be written in the form

$$
\frac{d^2W}{dz^2} + \left[ \frac{K_0 + K_1 z + K_2 z^2}{1-z^2} \right] \frac{dW}{dz} + \left[ \frac{K_3 + K_4 z}{1-z^2} \right] W = 0,
$$

(1) S. Persidès: in Memoriam Demetrios Eginitis (Athenes, 1975).