Spontaneous Symmetry Breaking in the $O(2)$ $\sigma$-Model (*).

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Summary. — We study, through the Schrödinger picture on lattice, the vacuum structure of the $O(2)$ $\sigma$-model. The analysis proceeds with an approximation which is an improvement of the usual perturbation method. We find in this work that no spontaneous symmetry breaking occurs, contrary to what is expected with the semi-classical approximation.

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1. – Introduction.

The phenomenon of spontaneous symmetry breaking (SSB)[1] is that the law of nature possesses some symmetry which is not respected by the vacuum state. The best known example is given by the $\phi^4$ theory with the «wrong sign» of mass term, as follows: Consider the Lagrangian density $\mathcal{L}$,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \mu^2 \phi^a \phi^a - \frac{\lambda}{4} (\phi^a \phi^a)^2,$$

where $\mu$ and $\lambda$ are the mass parameter and coupling constant, and $a = 1, 2, \ldots, n$ denotes the number of field components. For the energy of the system to be bounded below, $\lambda$ must be positive. We are interested in the case in which $\mu^2 < 0$. The phenomenon of SSB occurs when certain field operators acquire nonvanishing vacuum expectation values (v.e.v.), i.e.

$$\langle 0 | \phi^a | 0 \rangle \neq 0,$$

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or

\[ \langle 0 | \phi | 0 \rangle \neq 0, \quad \phi \equiv (\phi^a \phi^a)^{1/2}. \]

One difficulty in studying SSB in quantum field theory is the lack of a picture. Another one arises from the fact that the phenomenon of SSB is essentially nonperturbative, as the fields nearing the critical point highly fluctuate. Several models have been discussed in details \cite{2, 3}. The interesting point is that, whatever the semi-classical estimation is, the loop contribution to the potential might significantly alter the conclusions concerning the presence or absence of SSB.

This paper studies the vacuum structure of the \( O(2) \varphi^4 \) theory (\( \sigma \)-model). Through the Schrödinger picture on lattice \cite{4}, the analysis will proceed with a carefully designed series expansion. It is found in this work that no phenomena of SSB occur.

In sect. 2 we shall briefly describe the formalism used. Section 3 sets up the steps necessary for computing the v.e.v. of the scalar field in quantized theory. The results of calculations are shown in sect. 4. Discussions and remarks will be included in sect. 5. We will set \( c = 1 \) throughout this paper, but \( \hbar \) will be retained in all formulae so that the classical effect may be separated from the quantum one.

2. - The lattice formalism.

Consider the whole space (but time) as a large but finite square box with each side measured length \( \mathcal{L} \). For \( D \)-dimensional spatial box, the volume \( V \) is given by \( V = \mathcal{L}^N \). Let us divide it into \( N^N \) small cubes so that \( V = \mathcal{L}^N = (N \epsilon)^N \), where \( \epsilon \) is given by \( \epsilon = \mathcal{L}/N \). If \( \epsilon \) goes infinitesimal such that \( N \) goes infinite, we have a theory in the continuum. If \( N \epsilon \) goes infinite, the theory is defined in an infinite space. If \( V \) is otherwise defined, say depending on time, we have a theory in curve space-time. In this formalism, the fields at each point are replaced by constants on the corresponding lattice. For simplicity, the analysis will be illustrated in \( 1 + 1 \) dimensions, i.e. \( D = 1 \). The generalization to higher dimensions is straightforward whenever it is needed.

Let us go back to the \( O(2) \) \( \sigma \)-model, the Lagrangian density \( (1.1) \) with \( a = 1, 2 \). With \( D = 1 \) the total Lagrangian is given by

\[ L = \int \mathcal{L}' dz. \]

In the lattice formalism, we have

\[ \varphi'(z, t) \rightarrow \chi_j(t), \quad \varphi^2(z, t) \rightarrow y_j(t), \]

\[ \frac{\partial \varphi'}{\partial t} \rightarrow \frac{x_j - x_{j-1}}{\epsilon}, \quad \frac{\partial \varphi^2}{\partial t} \rightarrow \frac{y_j - y_{j-1}}{\epsilon}. \]

\[ \int dz \rightarrow \sum_{j=0}^{N-1} \epsilon \]