Remarks on the Local Regularity of the $\bar{\partial}$-Problem. (*)

FRANCO FAVILLI (**)
1. We use the definitions and notations given in [8], [10] and [6] by Kohn and Folland-Kohn.

Let $\Omega$ be a relatively compact domain in $\mathbb{C}^n$; we say that $\Omega$ has a smooth boundary $\partial \Omega$ if, in a neighborhood of $\partial \Omega$, we can define a smooth real-valued function $r$ such that

i) $r < 0$ in $\Omega$, $r = 0$ on $\partial \Omega$ and $r > 0$ outside $\Omega$

ii) $d^* r \neq 0$.

For each point $P \in \partial \Omega$, let us now consider the subspace of the complex tangent vectors

$$ T_{p,q}(\partial \Omega) = \left\{ L = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial z_i} : (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, L(r)(P) = 0 \right\}, $$

where $(z_1, \ldots, z_n) \in \mathbb{C}^n$; the hermitian form on $T_{p,q}(\partial \Omega)$ defined by

$$ \langle d^* r, L \rangle = \sum_{i,j=1}^{n} \frac{\partial^2 r}{\partial z_i \partial z_j} \xi_i \overline{\xi_j}, $$

is said the Levi-form of $r$ at $P$, where $L = \sum_{i=1}^{n} \xi_i (\partial / \partial z_i)$. We say that $\Omega$ is (weakly) pseudo-convex if the Levi-form is non-negative definite for each $P \in \partial \Omega$.

Let us consider now in $\Omega$ the inhomogeneous Cauchy-Riemann equation

$$ \partial^* u = \alpha, $$

where $\alpha$ is a $(p, q)$-form satisfying in $\Omega$ the compatibility condition $\partial^* \alpha = 0$. We recall that $\alpha$ is expressed as $\alpha = \sum_{I, J} \alpha_{I,J} dz^I \wedge d\bar{z}^J$ where $I$ and $J$ are strictly increasing sequences of positive integers of length $p$ and $q$, respectively, so that if, for example, $I = (i_1, \ldots, i_p)$ then $dz^I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$.

**Definition 1.** The $\partial$-problem is said to be locally regular in a point $P \in \partial \Omega$ if, for a neighborhood $U$ of $P$ and any $(p, q)$-form $\alpha$ on $\Omega$ with $\partial^* \alpha = 0$, there exists a solution $u$ ($u$ is a $(p, q-1)$-form) in $\Omega$ of the equation $\partial^* u = \alpha$ such that

$$ \text{sing supp } u \subset \text{sing supp } \alpha. $$

In [8] Kohn gave an example of a pseudo-convex domain in $\mathbb{C}^2$ where the $\partial$-problem is not locally regular on the boundary. Here we want to note that, as the $\partial$ operator is elliptic, the local regularity holds in $\Omega$ (see [6], for example), so that the real problem is on the boundary.

Starting from the above example it is possible to find a necessary condition for the local regularity of the $\partial$-problem in $\Omega$; precisely we have: