Dilatation and Conformal Invariance on Null Planes (*).

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Summary. — It is proven that a relativistic scalar field whose null-plane restriction exists ($Z^{-1} < \infty$), and which has $c$-number commutation relations there, transforms covariantly under dilatation and conformal transformations on a null plane. These transformations are thus unitarily implementable. The restriction of the conformal algebra to the null plane is discussed.

1. — Introduction.

Quantum field theory with respect to null planes rather than spacelike planes has aroused considerable interest in recent years. Most of this interest was related to the infinite-momentum limit of current commutators and to questions of scale invariance, especially in relation to deep inelastic scattering.

But a number of papers were concerned with null planes and null co-ordinate systems as a basis for a reformulation of field theory (1-2). Such a reformulation offers new and different mathematical aspects to a theory which is, of course, identical to the standard formulation in its physical predictions. For example, the commutator of a free scalar field in a spacelike plane vanishes and gives little useful information, but the field algebra on a null plane is non-Abelian and irreducible (1). Another difference lies in the unitary inequivalence of Fock representations of fields with different masses on a spacelike plane: there is unitary equivalence of representations of all masses on a null plane (1).

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These results suggest a study of symmetry properties on null planes. The unitary equivalence between zero-mass and finite-mass representations suggests that on null planes symmetries usually restricted to zero-mass fields may hold also for finite masses. The present paper will give precise mathematical meaning to this intuition.

The following notation will be employed (1). A null vector \( \mathbf{n} = n^\alpha = 0 \), in Minkowski space \( M_4 \) (metric trace \( = 2 \)) characterizes a family of parallel null planes to which \( \mathbf{n} \) is orthogonal. These planes are parametrized by the co-ordinate \( u \in (-\infty, \infty) \). The projection of four-vectors (such as \( \mathbf{p} \)) onto a null plane will be denoted by a subscript \( v \) (\( \mathbf{p}_v \equiv -\mathbf{n} \cdot \mathbf{p} \)). Another null vector \( \mathbf{m} = m^\alpha = 0 \), is required to be related to \( \mathbf{n} \) by \( \mathbf{m} \cdot \mathbf{n} = -1 \) and the corresponding projections are denoted by a subscript \( u \) (\( \mathbf{p}_u \equiv -\mathbf{m} \cdot \mathbf{p} \)). The remaining two-dimensional spacelike plane, orthogonal to \( \mathbf{m} \) and \( \mathbf{n} \) in \( M_4 \), is spanned by two unit vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \), such that

\[
e_1^2 = e_2^2 = 1, \quad e_1 \cdot e_2 = 0, \quad e_1 \cdot n = e_1 \cdot m = 0 \quad (i = 1, 2).
\]

The projections of \( \mathbf{p} \) on \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are \( \mathbf{p}_1 \equiv e_1 \cdot \mathbf{p}, \mathbf{p}_2 \equiv e_2 \cdot \mathbf{p} \).

We shall also use the notation

\[
\mathbf{p} \equiv (p_1, p_2), \quad \bar{\mathbf{p}} \equiv (p_1, p_2, p_3), \quad \text{d}^5\mathbf{p} \equiv \text{d}p_1 \text{d}p_2 \text{d}p_3 \text{d}p_4 \text{d}p_5,
\]

and in \( x \)-space

\[
x_u \equiv v, \quad x_v \equiv u, \quad x \equiv (x_1, x_2), \quad \bar{x} \equiv (x_1, x_2, v), \quad \text{d}^5x \equiv \text{d}x_1 \text{d}x_2 \text{d}x_3 \text{d}v.
\]

The scalar product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) is

\[
\mathbf{A} \cdot \mathbf{B} = A_\alpha B_\alpha = A_v B_u;
\]

thus

\[
\mathbf{p} \cdot x = p_1 x_1 - p_2 v - p_3 u.
\]

2. - The test function space.

It is well known (2) that the two-point function

\[
(\Omega, A(x) A(y) \Omega) \equiv \omega(x - y)
\]

(2) See, for example, S. S. Schweber: An Introduction to Relativistic Quantum Field Theory (Evanston, Ill., 1961).