Eigenfunctions and Eigenvalues of a Generalized Two-Mode Squeeze Operator.

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Summary. — In this paper we have constructed the continuum eigenfunctions and eigenvalue spectra for a generalized two-mode squeeze operator. Our analyses also show explicitly that proper eigenstates of the generalized two-mode squeeze operator do not exist, which implies that the generalized two-mode squeeze operator does not have a discrete spectrum.

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In a recent communication[1] Abdalla constructed the eigenfunctions and eigenvalue spectra for the two-mode squeeze operator \( S(x) = \exp[\alpha a_1^\dagger a_2^\dagger - \alpha^* a_1 a_2] \) as well as its generalization \( S(x, \beta, \gamma) = \exp[(\alpha a_1^{12} - \alpha^* a_1^2)/2 + (\beta a_2^{12} - \beta^* a_2^2)/2 + (\gamma a_1^2 a_2 - - \gamma^* a_1 a_2)] \), where \( a_i \) and \( a_i^\dagger \) are the usual boson annihilation and creation operators for the \( i \)-th field mode. Their results are, however, incorrect because of their apparent violations of the unitarity requirement. We believe that these problems crept into their formulation when they used the representation \( a_i^\dagger = x_i \) and \( a_i = \partial \partial x_i \) without taking care of the nonhermiticity of \( a_i \) and \( a_i^\dagger \). In fact, in a previous paper[2] we have shown that proper eigenstates of both single-mode and two-mode squeeze operators do not exist, which implies that the squeeze operators do not have a discrete spectrum. Using the Fock-Bargman representation[3, 4] we have also constructed the continuum eigenfunctions and eigenvalue spectra for the squeeze operators of both single and two modes. Now in this paper we shall extend our investigations to the generalized two-mode squeeze operator mentioned above. This generalized squeeze operator is related to the frequency converter model, which is described by the Hamiltonian \( H = \hbar \omega_1 a_1^\dagger a_1 + \hbar \omega_2 a_2^\dagger a_2 + k[a_1^\dagger a_2 \exp[-i\omega t] + a_1 a_2^\dagger \exp[i\omega t]] \).[1]

To begin with, let us consider the eigenvalue equation of the generalized two-mode squeeze operator

\[
S(x, \beta, \gamma) | \Psi \rangle = \exp[i \phi(x, \beta, \gamma)] | \Psi \rangle,
\]

where \( \phi(x, \beta, \gamma) \) is real. Since \( [S(x, \beta, \gamma), (\alpha a_1^{12} - \alpha^* a_1^2)/2 + (\beta a_2^{12} + \beta^* a_2^2)/2 + (\gamma a_1^2 a_2 - - \gamma^* a_1 a_2)] = \)

\[
0,
\]

where \( \phi(x, \beta, \gamma) \) is real. Since \( [S(x, \beta, \gamma), (\alpha a_1^{12} - \alpha^* a_1^2)/2 + (\beta a_2^{12} + \beta^* a_2^2)/2 + (\gamma a_1^2 a_2 - - \gamma^* a_1 a_2)] = \)

\[
0,
\]
\(-\gamma^* a_1 a_2) = 0\), one can simplify the eigenvalue equation as
\[
[(\alpha a_1^2 - \alpha^* a_1^2) + (\beta a_2^2 - \beta^* a_2^2) + 2(\gamma a_1^* a_2 - \gamma^* a_1 a_2)]|\Phi\rangle = i2\phi(x, \beta, \gamma)|\Phi\rangle.
\]

In the Hilbert space of entire functions of two complex variables, we have
\[
a_i^* \rightarrow z_i, \quad a_i \rightarrow \frac{\partial}{\partial z_i}, \quad |\Phi\rangle \rightarrow \Phi(z_1, z_2), \quad i = 1, 2,
\]
and the eigenvalue equation becomes
\[
\left[\left(\alpha z_1^2 - \alpha^* \frac{\partial^2}{\partial z_1^2}\right) + \left(\beta z_2^2 - \beta^* \frac{\partial^2}{\partial z_2^2}\right) + 2\left(\gamma z_1 z_2 - \gamma^* \frac{\partial}{\partial z_1 \partial z_2}\right)\right]\Phi(z_1, z_2) = i2\phi(x, \beta, \gamma)\Phi(z_1, z_2).
\]

Introducing the two new variables \(\omega\)
\[
\omega_+ = \sqrt{\frac{2A_+}{|A_+|}}(\mu z_1 + \nu z_2), \quad \omega_- = \sqrt{\frac{2A_-}{|A_-|}}(\mu^* z_1 - \nu^* z_2),
\]
where
\[
A_+ = 2\gamma \mu^* \nu^* + \alpha \mu^2 + \beta \nu^2, \quad A_- = 2\gamma \mu \nu - \alpha \nu^2 - \beta \mu^2, \quad |\mu|^2 + |\nu|^2 = 1,
\]
we can re-write eq. (4) as
\[
\left[|A_+|\left(\frac{\partial^2}{\partial \omega_+^2} + i\varepsilon_+ - \frac{\omega_+^2}{4}\right) - |A_-|\left(\frac{\partial^2}{\partial \omega_-^2} + i\varepsilon_- - \frac{\omega_-^2}{4}\right)\right]\Phi(\omega_+, \omega_-) = 0,
\]
with \(\phi(x, \beta, \gamma) = \varepsilon_+ |A_+| - \varepsilon_- |A_-|\), provided that
\[
\gamma = \frac{\alpha \mu^* \nu - \beta \mu \nu^*}{|\mu|^2 - |\nu|^2}.
\]

Here both \(\varepsilon_+\) and \(\varepsilon_-\) are real and independent of \(\alpha, \beta\) and \(\gamma\). It is obvious[2] that the eigenfunction takes the form of a product of two parabolic cylinder functions[5-7]:
\[
\Phi(z_1, z_2; \varepsilon_+, \varepsilon_-) = ND_{n_+}(\omega_+) D_{n_-}(\omega_-),
\]
where \(n_+ = -1/2 + i\varepsilon_+\), and \(N\) is a normalization factor. Strictly speaking, these functions do not belong to the Hilbert space because its norm diverges, \(i.e.[2]\)
\[
\int d^2 z_1 \int d^2 z_2 \exp[-|z_1|^2 - |z_2|^2]|\Phi(z_1, z_2; \varepsilon_+, \varepsilon_-)|^2 \rightarrow \infty.
\]

However, wave packets constructed from arbitrary linear superpositions of these functions do belong to the Hilbert space; in other words, they are continuum eigenfunctions with a continuous eigenvalue spectrum. The orthogonality property of