Separability of Schrödinger and Klein-Gordon Equations with a Vector Potential (*).

A. C. T. Wu

Department of Physics, University of Michigan - Ann Arbor, Mich.

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Summary. — Necessary and sufficient conditions are given for separability of the Schrödinger equation with a vector potential. The physical meaning of the condition on the vector potential is visualized with the aid of gauge invariance. The analysis can be carried out formally for the Klein-Gordon case.

1. – Introduction.

One of the most remarkable results in the separation of variables for partial differential equations were the works of STÄCKEL (1), ROBERTSON (2) and EISENHART (3-4). It is well known that the Hamilton-Jacobi equation is separable in the Stäckel sense. Forty years ago, ROBERTSON solved the problem of the Stäckel separability of the Schrödinger equation (with a scalar potential) in a Riemannian n-space, and about seven years later, EISENHART determined all these Stäckel systems in the Euclidean 3-space as well as the admissible forms of the potentials therein.

The purpose of this note is to settle the question of the Stäckel separability of the Schrödinger equation (as well as the Klein-Gordon equation) in the

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presence of a vector potential. Our result is essentially a negative statement. Namely, the requirement of the separation of variables imposes such a severe condition on the form of the vector potentials that, for all practical purposes, the equations are separable only with essentially trivial vector potentials. The physical meaning of the condition on the vector potential is visualized with the aid of gauge invariance.

In regard to the extension of Robertson’s analysis to the treatment of relativistic wave equations, such as the Klein-Gordon equation (c-number theory) in the presence of an electromagnetic potential, the following two remarks will suffice:

a) From the point of view of physics, since we are not interested in the exotic coordinate systems in which the time component is intricately entangled with the space components, it would be reasonable to consider only the stationary solutions of the form \( \psi(x) = \exp[-iEt] \varphi(x) \) and then proceed to discuss the separability in the space components. Formally this can be done. The condition on the vector potential for the Klein-Gordon equation is identical to that for the Schrödinger equation. (This is intuitively clear because of the common appearance of \((\partial - ieA)^2\) in both cases.) The separability conditions on the scalar potentials are however slightly different since the potential comes in the quadratic form \((E - e\varphi)^2\) in the KG case instead of the linear form \((E - e\varphi)\) in the Schrödinger case.

b) Strictly speaking, however, the question of the separability of relativistic wave equations in the sense described above remains only of pedagogical interest. Such separable solutions suffer from the fact that they are clearly not Lorentz-invariant solutions.

In the following, we shall therefore concentrate on the separability of the Schrödinger equation with a vector potential.

2. Background of the problem (c).

2.1. Hamilton-Jacobi equation. – In 1891 Stäckel (1) showed that the Hamilton-Jacobi equation

\[
\sum_{i=1}^{n} \frac{1}{f_{ii}} \left( \frac{\partial S}{\partial x_i} \right)^2 + V - \alpha = 0 ,
\]

where \( \alpha \) is a constant, admits separable solutions of the form \( S = \sum_i X_i(x_i) \), where \( X_i \) is a function of \( x_i \) alone, provided that i) given any \( n^2 \) functions

(\( c \)) For discussion on the separability on Helmholtz and Laplace equations, see, e.g. P. Moon and D. E. Spencer: Journ. Franklin Inst., 253, 585 (1952).