Sampling and Π-sampling expansions

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Abstract. Using the hyperfinite representation of functions and generalized functions this paper develops a rigorous version of the so-called ‘delta method’ approach to sampling theory. This yields a slightly more general version of the classical WKS sampling theorem for band-limited functions.

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1. Preliminaries

The classical sampling expansion for band-limited functions can be derived rigorously by several distinct arguments, but the use of so-called ‘delta-methods’ offers an approach which is intuitively most satisfying. A rigorous form of a delta-method derivation of the sampling expansion has been presented, using standard analysis, by Nashed and Walter [1]. In this paper we consider instead a non-standard approach to sampling theory. The hyperfinite representation of functions and generalized functions has been studied in an earlier paper [2], and the same notation and conventions will be used here. In particular, $\kappa \in \mathbb{N}_\infty$ denotes a given even infinite hypernatural number, $\varepsilon = \kappa^{-1} \approx 0$ and

$$\Pi \equiv \Pi_\kappa = \left\{ -\frac{\kappa}{2}, -\frac{\kappa}{2} + \varepsilon, \ldots, 0, \ldots, \frac{\kappa}{2} - \varepsilon \right\}$$

$$= \left\{ \left( -\frac{\kappa^2}{2} + j \right) \varepsilon : j = 1, 2, \ldots, \kappa^2 \right\} \subset \mathbb{R}$$

is the (unbounded) hyperfinite line. Given a standard point $r \in \mathbb{R}$, define the $\Pi$-monad of $r$ by

$$\text{mon}_\Pi(r) = \text{st}^{-1}_\Pi(r) = \text{mon}(r) \cap \Pi$$

where ‘mon’ denotes the usual monad of a standard number in $\mathbb{R}$. Then the set $\Pi_b = \bigcup_{r \in \mathbb{R}} \text{mon}_\Pi(r) = \text{st}^{-1}_\Pi(\mathbb{R}) \subset \Pi$ is the nearstandard hyperfinite line and $\Pi_\infty = \Pi \setminus \Pi_b$ is the set of remote points of the hyperfinite line. For every subset $A \subset \mathbb{R}$ define $^*A_\Pi = ^*A \cap \Pi$ and $\text{ns}_\Pi(^*A) = ^*A \cap \Pi_b = \bigcup_{a \in A} \text{mon}_\Pi(a)$.

By $F_\Pi$ we denote the algebra of all internal functions $F : \Pi \to ^*\mathbb{C}$ which are periodically extended to the infinite grid $\varepsilon \cdot \mathbb{Z}$. The two difference operators $D_+, D_- : F_\Pi \to F_\Pi$ defined, for every function $F$ and $x \in \Pi$, by

$$D_+ F(x) = \varepsilon^{-1} [F(x + \varepsilon) - F(x)] \quad \text{and} \quad D_- F(x) = \varepsilon^{-1} [F(x) - F(x - \varepsilon)]$$

are called, respectively, the forward and the backward $\Pi$-difference operators (of first order). Iterating $D_+$ (or $D_-$) we obtain higher order $\Pi$-difference operators: for every
(finite or infinite) \( n \in ^* \mathbb{N}_0 \)

\[
\mathbf{D}^n F(x) = \mathbf{D}_+ (\mathbf{D}^{n-1}_+ F(x)), \quad x \in \Pi
\]

and similarly for \( \mathbf{D}^n \). It is easily seen that for any two functions \( F, G \in \mathbb{F}_\Pi \) we have, (both for \( \mathbf{D}_+ \) and \( \mathbf{D}_- \)),

\[
\mathbf{D} (F + G) = \mathbf{D} F + \mathbf{D} G \quad \text{and} \quad \mathbf{D} (F \cdot G) = (\mathbf{D} F) G + F (\mathbf{D} G) \pm \varepsilon (\mathbf{D} F) (\mathbf{D} G),
\]

where we take \( \pm \varepsilon \) according to the use of \( \mathbf{D}_+ \) or \( \mathbf{D}_- \), respectively.

For every \( \alpha, x \in \Pi \) define the \( \Pi \)-intervals (containing only points in \( \Pi \)) \( J^+_\alpha(x) \) and \( J^-_\alpha(x) \) as follows:

\[
J^+_\alpha(x) = \begin{cases} 
(x, \alpha]_\Pi & \text{if } x < \alpha \\
[x, \alpha)_\Pi & \text{if } x > \alpha 
\end{cases}
\]

\[
J^-_\alpha(x) = \begin{cases} 
[\alpha, x)_\Pi & \text{if } x < \alpha \\
(\alpha, x)_\Pi & \text{if } x > \alpha 
\end{cases}
\]

while for \( x = \alpha \) we have \( J^+_\alpha(x) = \emptyset = J^-_\alpha(x) \). For any \( F \in \mathbb{F}_\Pi \) define the functions \( S_+ F \) and \( S_- F \) to be the forward and backward \( \Pi \)-sums of \( F \) which are zero at the origin and which, for every \( x \in \Pi \setminus \{0\} \) are defined by

\[
S^+_\alpha F(x) = \sum_{t \in J^+_\alpha(x)} \varepsilon F(t) \quad \text{and} \quad S^-_\alpha F(x) = \sum_{t \in J^-_\alpha(x)} \varepsilon F(t).
\]

The \( \Pi \)-sum operators \( S_+ \) and \( S_- \) both transform \( \mathbb{F}_\Pi \) into \( \mathbb{F}_\Pi \). Moreover, for every \( F \in \mathbb{F}_\Pi \), we have

\[
\mathbf{D}_+ S_+ F = F \quad \text{and} \quad \mathbf{D}_- S_- F = F
\]

that is, \( S_+ \) and \( S_- \) are left inverses for \( \mathbf{D}_+ \) and \( \mathbf{D}_- \), respectively.

Define the translation operator \( \tau_\alpha : \mathbb{F}_\Pi \to \mathbb{F}_\Pi \) (with \( \alpha \in \Pi \)) by setting \( \tau_\alpha F(x) = F(x - \alpha) \) for every function \( F \) and \( x \in \Pi \).

2. \( \Pi \)-periodic functions and \( \Pi \)-Fourier sums

2.1 \( \Pi \)-Fourier sums

For any internal function \( F \in \mathbb{F}_\Pi \) define the \( \Pi \)-periodic transform of \( F \) with period 1 (or, simply, the \( \Pi \)-periodic transform\(^1\) of \( F \)) to be the internal function \( \mathbf{T}_\Pi[F] \) in \( \mathbb{F}_\Pi \) which is such that

\[
\mathbf{T}_\Pi[F](x) = \sum_{n \in ^* \mathbb{Z}_\Pi} F(x - n), \quad x \in \Pi
\]

where \( ^* \mathbb{Z}_\Pi \equiv ^* \mathbb{Z} \cap \Pi \). (As usual we suppose that the function \( \mathbf{T}_\Pi[F] \) is periodically extended to the whole of the discrete line \( \varepsilon ^* \mathbb{Z} \).) In particular for the function \( \Delta_0 \) defined by

\[
\Delta_0(x) = \begin{cases} 
\kappa & \text{if } x = 0 \\
0 & \text{if } x \neq 0 
\end{cases}
\]

\(^1\)Unless explicitly stated, \( \Pi \)-periodic transforms will always be understood here to have period 1.