Periodic and boundary value problems for second order differential equations

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Abstract. In this paper we study second order scalar differential equations with Sturm–Liouville and periodic boundary conditions. The vector field \( f(t, x, y) \) is Caratheodory and in some instances the continuity condition on \( x \) or \( y \) is replaced by a monotonicity type hypothesis. Using the method of upper and lower solutions as well as truncation and penalization techniques, we show the existence of solutions and extremal solutions in the order interval determined by the upper and lower solutions. Also we establish some properties of the solutions and of the set they form.

Keywords. Upper solution; lower solution; order interval; truncation map; penalty function; Caratheodory function; Sobolev space; compact embedding; Dunford–Pettis theorem; Arzela–Ascoli theorem; extremal solution; periodic problem; Sturm–Liouville boundary conditions.

1. Introduction

The method of upper and lower solutions offers a powerful tool to establish the existence of multiple solutions for initial and boundary value problems of the first and second order. This method generates solutions of the problem, located in an order interval with the upper and lower solutions serving as bounds. In fact the method is often coupled with a monotone iterative technique which provides a constructive way (amenable to numerical treatment) to generate the extremal solutions within the order interval determined by the upper and lower solutions.

In this paper we employ this technique to study scalar nonlinear periodic and boundary value problems. The overwhelming majority of the works in this direction, assume that the vector field is continuous in all variables and they look for solutions in the space \( C^2(0, b) \). We refer to the books by Bernfeld–Lakshmikantham [2] and Gaines–Mawhin [6] and the references therein. The corresponding theory for discontinuous (at least in the time variable \( t \)) nonlinear differential equations is lagging behind. It is the aim of this paper to contribute in the development of the theory in this direction. Dealing with discontinuous problems, leads to Caratheodory or monotonicity conditions and to Sobolev spaces of functions of one variable. It is within such a framework that we will conduct our investigation in this paper. We should mention that an analogous study for first order problems can be found in Nkashama [18].
2. Sturm–Liouville problems

Let \( T = [0, b] \). We start by considering the following second order boundary value problem:

\[
\begin{aligned}
-
& x''(t) = f(t, x(t), x'(t)) \quad \text{a.e. on } T \\
& (B_0x)(0) = \nu_0, \quad (B_1x)(b) = \nu_1
\end{aligned}
\]  

(1)

Here \((B_0x)(0) = a_0x(0) - c_0x'(0)\) and \((B_1x)(b) = a_1x(b) + c_1x'(b)\), with \(a_0, c_0, a_1, c_1 \geq 0\) and \(a_0(a_1b + c_1) + c_0a_1 \neq 0\). Note that if \(c_0 = c_1 = \nu_0 = \nu_1 = 0\), then we have the Dirichlet (or Picard in the terminology of Gaines–Mawhin [6]) problem. The vector field \(f(t, x, y)\) is not continuous, but only a Caratheodory function; i.e., it is measurable in \( t \in T \) and continuous in \((x, y) \in \mathbb{R} \times \mathbb{R}\) (later the continuity in \(y\) will be replaced by a monotonicity condition). Hence \(x''(\cdot)\) is not continuous, but only an \(L^1(T)\)-function. Recently Nieto–Cabada [17] considered a special case of (1) with \(f\) independent of \(y\). Also there is the work of Omari [19] where \(f\) is continuous.

We will be using the Sobolev spaces \(W^{1,1}(T)\) and \(W^{2,1}(T)\). It is well known (see for example Brezis [3], p. 125), that \(W^{1,1}(T)\) is the space of absolutely continuous functions and \(W^{2,1}(T)\) is the space of absolutely continuous function whose derivative is absolutely continuous too.

**DEFINITION**

A function \(\psi \in W^{2,1}(T)\) is said to be a ‘lower solution’ for problem (1) if

\[
\begin{aligned}
-
& -\psi''(t) \leq f(t, \psi(t), \psi'(t)) \quad \text{a.e. on } T \\
& (B_0\psi)(0) \leq \nu_0, \quad (B_1\psi)(b) \leq \nu_1
\end{aligned}
\]  

(2)

A function \(\phi \in W^{2,1}(T)\) is said to be an ‘upper solution’ for problem (1) if the inequalities in (2) are reversed.

For the first existence theorem we will need the following hypotheses:

\(H(f)\):

\(f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a function such that

(i) for every \(x, y \in \mathbb{R}\), \(t \to f(t, x, y)\) is measurable;

(ii) for every \(t \in T\), \((x, y) \to f(t, x, y)\) is continuous;

(iii) for every \(r > 0\) there exists \(\gamma_r \in L^1(T)\) such that \(|f(t, x, y)| \leq \gamma_r(t)\) a.e. on \(T\) for all \(x, y \in \mathbb{R}\) with \(|x|, |y| \leq r\).

\(H_0\): There exists an upper solution \(\phi\) and a lower solution \(\psi\) such that \(\psi(t) \leq \phi(t)\) for every \(t \in T\) and there exists \(h \in C(\mathbb{R}_+, (0, \infty))\) such that \(|f(t, x, y)| \leq h(|y|)\) for all \(t \in T\) and all \(x, y \in \mathbb{R}\) with \(\psi(t) \leq x \leq \phi(t)\) and

\[
\int_b^x \frac{dr}{h(r)} > \max_{t \in T} \phi(t) - \min_{t \in T} \psi(t), \quad \text{with} \quad \lambda = \max_{t \in T} \left\{\psi(0) - \phi(0)\right\}.
\]

**Remark.** The second part of hypothesis \(H_0\) (the growth condition on \(f\)), is known as the ‘Nagumo growth condition’ and guarantees an \(a \ priori\) \(L^\infty\)-bound for \(x'(\cdot)\). More precisely, if \(H_0\) holds, then there exists \(N_1 > 0\) (depending only on \(\phi, \psi, h\)) such that for every \(x \in W^{2,1}(T)\) solution of \(-x''(t) = f(t, x(t), x'(t))\) a.e. on \(T\) with \(\psi(t) \leq x(t) \leq \phi(t)\) for all \(t \in T\), we have \(|x'(t)| \leq N_1\) for all \(t \in T\) (the proof of this, is the same (with minor modifications) with that of Lemma 1.4.1, p. 26 of Bernfeld–Lakshmikantham [2]).