Subanalytic bundles and tubular neighbourhoods of zero-loci

VISHWAMBHAR PATI
Stat-Math Unit, Indian Statistical Institute, R.V. College Post, Bangalore 560 059, India
E-mail: pati@isibang.ac.in
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Abstract. We introduce the natural and fairly general notion of a subanalytic bundle (with a finite dimensional vector space $P$ of sections) on a subanalytic subset $X$ of a real analytic manifold $M$, and prove that when $M$ is compact, there is a Baire subset $U$ of sections in $P$ whose zero-loci in $X$ have tubular neighbourhoods, homeomorphic to the restriction of the given bundle to these zero-loci.

Keywords. Subanalytic set; subanalytic bundle; Strong Whitney stratification; Verdier stratification; tubular neighbourhood; zero-locus of subanalytic bundle; stratified transversality.

1. Introduction

In this paper, we introduce the notion of a subanalytic bundle $E$ (generated by a finite dimensional space $P$ of global sections) on a (not necessarily closed) subanalytic set $X$ inside a real analytic manifold $M$, as a natural generalisation of real analytic bundles on real analytic spaces to the subanalytic setting. We prove (in Theorem 6.6 below) that for $M$ compact, there exists a Baire subset $U$ of sections in $P$, such that for $s \in U$, there exist tubular neighbourhoods of the zero-locus $Z = s^{-1}(0_{E})$ of $s$ in $X$, i.e. which are homeomorphic to the restriction of the given bundle to $Z$. To keep the account self-contained we recall basic facts about subanalytic sets in §2 and Strong Whitney (SW) stratifications (defined by Verdier) in §4.

We remark here that the main Theorem 6.6 would follow from Theorem 1.11 on p. 48 of [G-M]. However, the proof (‘deformation to the normal bundle’) sketched in [G-M] is incomplete, at least in the generality that it is stated. In this generality, the stratified submersion they construct is not proper (as was pointed out by V Srinivas), and hence Thom’s First Isotopy Lemma is inapplicable. To circumvent this, we have imposed the hypothesis of compactness on the ambient real analytic manifold $M$ containing the subanalytic set $X$, but no compactness assumption on $X$. Our hypotheses are general enough to cover most situations arising in real or complex algebraic geometry (see Example 2.2 and Remark 6.7).

2. Subanalytic sets and maps

Let $M$ be a real-analytic manifold. We will always assume $M$ to be connected, Hausdorff, second countable and paracompact.
DEFINITION 2.1
We say \( X \subset M \) is a subanalytic set of \( M \) if there exists an open covering \( \mathcal{U} \) of \( M \) (not just of \( X \)) such that for each \( U \in \mathcal{U} \),
\[
X \cap U = \bigcup_{i=1}^{p} (f_{ij}(A_{ij}) - f_{j2}(A_{j2})),
\]
where \( f_{ij}: N_{ij} \to U \) for \( 1 \leq i \leq p \) and \( j = 1, 2 \), are real analytic maps of real analytic manifolds \( N_{ij} \), \( A_{ij} \) are closed analytic subsets of \( N_{ij} \) and \( f_{ij|A_{ij}} \) are proper maps (see Proposition 3.13 in [B-M] and Definition 3.1 in [Hi]).

Example 2.2. All real (resp. complex) analytic subsets of a real (resp. complex) analytic manifold are subanalytic sets. In particular, (real or complex) algebraic subsets of a (real or complex) algebraic manifold (such as projective space, or Grassmannians) are subanalytic sets. Also since subanalytic subsets of a real analytic manifold form a Boolean algebra (see (i) of Proposition 2.7 below) all real (resp. complex) analytically (or algebraically) constructible sets in a real (resp. complex) analytic (or algebraic) manifold are subanalytic sets. In particular, all (real or complex) affine algebraic varieties are subanalytic in both affine space, and projective space. Real or complex quasiprojective varieties are subanalytic sets in the corresponding projective spaces.

Remark 2.3. A real analytic subset \( X \) (subspace) of a real analytic manifold \( M \) is a closed subset of \( M \) by definition. In particular if \( M \) is compact, so is \( X \). By contrast, a subanalytic set \( X \) of a real analytic manifold \( M \) need not be closed, and need not be compact even if \( M \) is compact.

DEFINITION 2.4
Let \( X \subset M \) and \( Y \subset N \) be subanalytic sets in the real analytic manifolds \( M, N \) respectively. We say that a map \( f: (X, M) \to (Y, N) \) is a subanalytic map if \( f: X \to Y \) is a continuous map, and the graph
\[
\Gamma_f := \{(x, y) \in M \times N : x \in X, \ y = f(x)\}
\]
is a subanalytic set in \( M \times N \) (see [Ha], 4.1, or Definition 3.2 in [B-M]). Note that although the map \( f \) is defined only on \( X \), its subanalyticity depends on the ambient \( M, N \), as we shall see in Remark 2.6 below.

Notation 2.5. If \( X, M, N \) are as above, and \( f: (X, M) \to (N, N) \) is a subanalytic map, we shall write \( f: (X, M) \to N \) is a subanalytic map, for notational convenience.

Remark 2.6. The subanalyticity (or analyticity) of a set, or of a map depends on the ambient spaces \( M, N \). For example, \( X = \{\frac{1}{n}\}_{n \in \mathbb{N}} \) is a subanalytic set in \((0, \infty)\), but not in \( \mathbb{R} \). In the former, it is the zero set of the analytic function \( \sin \frac{\pi}{x} \), so analytic and hence subanalytic in \((0, \infty)\). It is not subanalytic in \( \mathbb{R} \) because the connected components of its germ at 0 in \( \mathbb{R} \) do not form a locally finite collection (see (viii) of Proposition 2.7 below).

Similarly, the map \((0, 1), (0, 1)) \to (\mathbb{R}, \mathbb{R})\) defined by \( x \mapsto \sin \frac{\pi}{x} \) is clearly subanalytic, because its graph \( \Gamma := \{(x, \sin \frac{\pi}{x}) : x \in (0, 1)\}\) is an analytic (hence subanalytic) subset of \((0, 1) \times \mathbb{R}\). On the other hand, the same mapping regarded as a map \((0, 1), \mathbb{R}) \to (\mathbb{R}, \mathbb{R})\) is not subanalytic, since \( \Gamma \) is not a subanalytic subset in \( \mathbb{R} \times \mathbb{R} \).