Corners of normal matrices

RAJENDRA BHATIA and MAN-DUEN CHOI∗

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,
New Delhi 110 016, India
∗Department of Mathematics, University of Toronto, Toronto M5S 2E4, Canada
E-mail: rbh@isid.ac.in; choi@math.toronto.edu

To Kalyan Sinha on his sixtieth birthday

Abstract. We study various conditions on matrices $B$ and $C$ under which they can be the off-diagonal blocks of a partitioned normal matrix.

Keywords. Normal matrix; unitary matrix; norm; completion problem; dilation.

The structure of general normal matrices is far more complicated than that of two special kinds — hermitian and unitary. There are many interesting theorems for hermitian and unitary matrices whose extensions to arbitrary normal matrices have proved to be extremely recalcitrant (see e.g., [1]). The problem whose study we initiate in this note is another one of this sort.

We consider normal matrices $N$ of size $2n$, partitioned into blocks of size $n$ as

$$N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1}$$

Normality imposes some restrictions on the blocks. One such restriction is the equality

$$\|B\|_2 = \|C\|_2 \tag{2}$$

between the Hilbert–Schmidt (Frobenius) norms of the off-diagonal blocks $B$ and $C$. If $T$ is any $m \times m$ matrix with entries $t_{ij}$, then

$$\|T\|_2 = \left(\sum_{j=1}^{m} |t_{ij}|^2 \right)^{1/2}.$$

The equality (2) is a consequence of the fact that the Euclidean norm of the $j$th column of a normal matrix is equal to the Euclidean norm of its $j$th row.

Replacing the Hilbert–Schmidt norm by another unitarily invariant norm, we may ask whether the equality (2) is replaced by interesting inequalities. Let $s_1(T) \geq \cdots \geq s_m(T)$ be the singular values of $T$. Every unitarily invariant norm $\|||T|||$ is a symmetric gauge function of $\{s_j(T)\}$ (see chapter IV of [1] for properties of such norms). Much of our concern in this note is with the special norms

$$\|T\|_2 = (\text{tr} \ T^*T)^{1/2} = \left(\sum_{j=1}^{m} s_j^2(T) \right)^{1/2}.$$
and
\[ \|T\| = s_1(T) = \sup_{x \in \mathbb{C}^m, \|x\|=1} \|Tx\|. \] (3)
The latter is the norm of \(T\) as a linear operator on the Euclidean space \(\mathbb{C}^m\). Clearly
\[ \|T\| \leq \|T\|_2 \leq \sqrt{m} \|T\|, \] (4)
for every \(m \times m\) matrix \(T\).

If the matrix \(N\) in (1) is hermitian, then \(C = B^*\), and hence, \(|||C||| = |||B|||\) for all unitarily invariant norms. If \(N\) is unitary, then \(AA^* + BB^* = A^*A + C^*C = I\). Hence, the eigenvalues \(\lambda_j\) satisfy the relations
\[ \lambda_j(BB^*) = \lambda_j(I - AA^*) = 1 - \lambda_j(AA^*) \]
\[ = 1 - \lambda_j(A^*A) = \lambda_j(I - A^*A) = \lambda_j(C^*C). \]
Thus \(B\) and \(C\) have the same singular values, and again \(|||B||| = |||C|||\) for all unitarily invariant norms.

This equality of norms does not persist when we go to arbitrary normal matrices, as we will soon see. From (2) and (4) we get a simple inequality
\[ |||B||| \leq \sqrt{n} |||C|||. \] (5)
One may ask whether the two sides of (5) can be equal, and that is the first issue addressed in this note.

When \(n = 2\), it is not too difficult to construct a normal matrix \(N\) of the form (1) in which \(\|B\| = \sqrt{2}\|C\|\). One example of such a matrix is
\[
N = \begin{bmatrix}
    0 & 0 & \sqrt{2} & 0 \\
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 1 \\
    1 & 0 & 0 & 0
\end{bmatrix}.
\] (6)
When \(n = 3\), examples seem harder to come by. One that preserves some of the features of (6) is given by the matrix
\[
N = \begin{bmatrix}
    0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 & \sqrt{3} & 0 & 0 \\
    0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 & 0 & 0 \\
    \sqrt{\frac{2}{\sqrt{3}} + 1} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & \sqrt{\frac{2}{\sqrt{3}} - 1} & 0 \\
    0 & 1 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} & 0 \\
    1 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{\sqrt{3}}} 
\end{bmatrix}.
\] (7)