Local zeta functions of general quadratic polynomials*

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Dedicated to the memory of Professor K G Ramanathan

Abstract. This paper is concerned with the kind of local zeta functions now often called Igusa's local zeta functions: A simple closed form of such a zeta function for an arbitrary quadratic form, its variants, and an application are given.

Keywords. Local zeta function; quadratic polynomials.

1. Introduction

The local zeta functions which we shall consider in this paper are of the form

\[ Z(s) = \int_L |f(x)|_K^s \, dx, \quad \Re(s) > 0, \]

where \( L \) is a lattice in a vector space \( V \) over a \( p \)-adic field \( K \), \(| \cdot |_K \) is an absolute value on \( K \), \( f \) is a polynomial function on \( V \), \( s \) is a complex variable, and \( dx \) is the Haar measure on \( V \) normalized as \( Z(0) = 1 \). If \( q \) is the cardinality of the residue class field of \( K \) and if \( \text{char} \ (K) = 0 \), our general theorem states that \( Z(s) \) is a rational function of \( q^{-s} \); cf. [2]. Furthermore \( Z(s) \) has been computed in a large number of cases. We might add that in some cases computations are very difficult and that no algorithm to compute \( Z(s) \) for a general \( f \) is known.

Now, in the difficult cases where \( Z(s) \) has been computed, one has often encountered auxiliary integrals of the above form where \( f \) is a quadratic polynomial with coefficients depending on some parameters. In this paper we shall give a simple closed form to \( Z(s) \) in the case where \( f \) is an arbitrary nondegenerate quadratic form and also in some inhomogeneous cases; for the sake of simplicity we have assumed that \( q \) is odd. The formula has turned out to be useful for the computation of \( Z(s) \) in some new cases and for the simplification in some known cases. We have included just one application at the end of the paper.

2. Preliminaries

As in the Introduction, we denote by \( V \) a (finite dimensional) vector space over a \( p \)-adic field \( K \) and by \( L \) a lattice in \( V \). If \( S \) is any subset of \( V \), we shall denote by \( KS \) the \( K \)-span of \( S \) in \( V \). If \( O_K \) is the maximal compact subring of \( K \), then \( L \) is a free
$O_K$-submodule of $V$ such that $KL = V$. We shall denote by $n(L)$ the rank of $L$, i.e., $n(L) = \dim(KL)$. We choose $\pi$ from $O_K$ such that $\pi O_K$ becomes the maximal ideal of $O_K$ and denote by $q$ the cardinality of the residue class field $O_K/\pi O_K$, i.e., $O_K/\pi O_K = F_q$.

In general if $R$ is any associative ring with 1, we shall denote by $R^*$ the group of units of $R$. We shall assume, once and for all, that $q$ is odd; then

$$(O_K^*)/(O_K^*)^2 \cong F_q^*/(F_q^*)^2 \cong \{-1\}.$$

We shall denote the corresponding homomorphism of $O_K^*$ to $\{-1\}$ by $\chi$.

We take a quadratic form $Q$ on $V$, i.e., a $K$-valued function on $V$ satisfying $Q(ax) = a^2 Q(x)$ for every $a$ in $K$ and $x$ in $V$ such that $Q(x, y) = Q(x + y) - Q(x) - Q(y)$ is $K$-bilinear in $x, y$. If $\{w_1, \ldots, w_n\}$ is a $K$-basis for $V$ so that $n = \dim(V)$, then

$$Q \left( \sum_{1 \leq i \leq n} x_i w_i \right) = \sum_{1 \leq i \leq n} Q(w_i) x_i^2 + \sum_{i < j} Q(w_i, w_j) x_i x_j$$

for every $x_1, \ldots, x_n$ in $K$. Therefore if $\{w_1, \ldots, w_n\}$ is an $O_K$-basis for $L$, then $Q$ is $O_K$-valued on $L$ if and only if the coefficients of the above homogeneous polynomial of degree 2 in $x_1, \ldots, x_n$ are all in $O_K$. We shall assume that $Q$ is nondegenerate, i.e., that $Q(x, y) = 0$ for all $y$ in $V$ implies $x = 0$. If $\{w_1, \ldots, w_n\}$ is an $O_K$-basis for $L$ and if $h$ is the square matrix of degree $n$ with $Q(w_i, w_j)$ as its $(i, j)$-entry for $1 \leq i, j \leq n$, then $\det(h) \neq 0$. We define the discriminant $D(L, Q)$ of $(L, Q)$ as

$$D(L, Q) = (-1)^{n(n-1)/2} \det(h).$$

We observe that $D(L, Q)(O_K^*)^2$ is independent of the choice of the $O_K$-basis for $L$.

We call $(L, Q)$ unimodular if $Q$ is $O_K$-valued on $L$ and if $D(L, Q)$ is in $O_K^*$. In the general case $(L, Q)$ has the following Jordan decomposition:

There exist $O_K$-submodules $L_1, \ldots, L_t$ of $L$ with $L$ as their sum and a sequence of integers $e_1 < \cdots < e_t$ such that $KL_1, \ldots, KL_t$ are mutually orthogonal with respect to $Q(x, y)$ and $(L_i, Q_i)$, where $Q_i = \pi^{-e_i} Q|_{KL_i}$, is unimodular for $1 \leq i \leq t$. Furthermore if we put $n_i = n(L_i)$, then

$$\text{inv}(L, Q) = \{n_i, e_i, \chi(D(L_i, Q_i)), 1 \leq i \leq t\}$$

characterizes the isomorphism class of $(L, Q)$.

We refer to O'Meara [5], Chapter IX for the theory of Jordan decompositions. We keep in mind that $M = L_{j_1} + \cdots + L_{j_t}$, where $1 \leq j_1 < \cdots < j_t \leq t$, gives the Jordan decomposition of $(M, Q|KM)$.

### 3. $Z(s)$ for a quadratic form

We start from the Jordan decomposition of $(L, Q)$ recalled in the previous section. If we define $Q^0$ as $Q_1 + \cdots + Q_t$, i.e., as

$$Q^0(x_1 + \cdots + x_t) = Q_1(x_1) + \cdots + Q_t(x_t)$$

for all $x_1, \ldots, x_t$ respectively in $KL_1, \ldots, KL_t$, then $(L, Q^0)$ is unimodular. We put