A Formula of General Solution for a Class of Homogeneous Trinomial Recurrence of Variable Coefficients with Two Indices

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Abstract: In practice, it is very difficult to find the solution of recurrence relation by using the characteristic roots. By applying iteration and induction we present an explicit formula of general solution for a class of homogeneous trinomial recurrence of variable coefficients with two indices. It provides a concrete and applicable model to solve the relevant problem with computer.

Key words: a formula of general solution; homogeneous trinomial recurrence; variable coefficients; two indices

CLC number: O 157.1

0 Introduction

As we know, the success of solving any linear homogeneous recurrence relation with constant coefficients depends on finding the roots of the characteristic equation, and this may not always be possible. Once the roots are known, in order to satisfy the initial values it is necessary to solve a system of linear equations. If the order of the recurrence relation is \( k \), there are \( k \) equations in unknown. Thus in practice, it may be very difficult, if not impossible, to find the solution of the recurrence relation by using the characteristic roots. If a recurrence relation is not linear and homogeneous with constant coefficients, no methods have been given that will enable one to find a solution. In this paper, according to the principle of solving algebraic equation, we present the formula of general solution for a class of homogeneous recurrence of variable coefficients with two indices by applying iteration and induction. It provides a concrete model to solve the relevant problems by modern computing tools.

For simplification, we denote

\[
F(i,j;M,L) = \sum_{r=0}^{[\frac{M}{p}]} \sum_{N_{r-1}} \left\{ \prod_{m=1}^{r} q(i - (k_m - 1) - p(m - 1) - N_{m-1} \cdot j - (k_m - 1) - p(m - 1) - N_{m-1}) \right\} \prod_{\lambda=1}^{r} q(i - p(\lambda - 1) - N_{\lambda} \cdot j - p(\lambda - 1) - N_{\lambda})
\]

Received date: 2004-10-05
Foundation item: Supported by the National Natural Science Foundation of China (10071059)
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where \(i, j, M\) and \(L\) are non-negative integers, \(p\) is an integer bigger than 1.

Put \(N_r = n_1 + n_2 + \cdots + n_r, (n_1, n_2, \ldots, n_r, (r \geq 1)\) are non-negative integers); denote

\[
\sum_{i=0}^{L} = \sum_{n_1=0}^{L-n_1} \cdots \sum_{n_r=0}^{L-n_r} 1, \quad L \geq 0
\]

where \(r\) is a positive integer; when \(r = 1\), put \(n_r = N_r = 0\) and \(\sum_{i=0}^{L} = 1 (L \geq 0)\). Denote \(\prod_{m=1}^{r} f(n_m) = f(n_1) f(n_2) \cdots f(n_r)\); if \(r = 1\), then put \(\prod_{m=1}^{r} f(n_m) = 1\). \([y]\) is the biggest integer which is no bigger than \(y\), where \(y\) is a real.

1 Main Result

**Theorem 1** The homogeneous trinomial recurrence of variable coefficients with two indices

(A) \(s(i, j) = \varphi(i, j) s(i-1, j-1) + \psi(i, j) s(i-p, j-p), i, j > p\)

(B) \(s(i, T) = c(i, T), T = 0, 1, \ldots, p-1; i \geq T\)

(C) \(s(i, j) = 0, i < 0, j < 0\) or when \(i = 0, 1, \ldots, p-1, i < j\)

has the general solution

\[
s(i, j) = \sum_{t=0}^{2(p-1)} \{\varphi(i-j+t, t) \{F(i, j; j-t, j-t-\varphi) c(i-j+t-p, t-p)\}}
\]

\[
+ F(i, j; j-(p-1), j-(p-1)-\varphi) c(i-j+(p-1), p-1)\}, \quad i > 0
\]

where \(i\) and \(j\) are non-negative integers, \(p\) is an integer bigger than 1, \(\varphi(i, j)\) and \(\psi(i, j)\) are real function of \(i\) and \(j\), \(c(i, T) (i = 0, 1, \ldots, T, T = 0, 1, \ldots, p-1)\) is real number.

**Proof** We use 2-uple induction principle\(^{1,5,6}\) (for \(p \geq 2\)).

First, when \(j = k (0 \leq k \leq p-1)\), Eq. (5) is valid for every non-negative integer \(i\); so we can suppose when \(j \leq j-1\), especially \(j = p\) and \(j-1\), where \(J\) is a constant, (5) is valid for every non-negative integer \(i\). Now we prove (5) is valid under the condition that \(j = J\) and for every non-negative integer \(i\), that is to prove

\[
s(i, J) = \sum_{t=0}^{2(p-1)} \{\varphi(i-J+t, t) \{F(i, J; J-t, J-t-\varphi) c(i-J+t-p, t-p)\}}
\]

\[
+ F(i, J; J-(p-1), J-(p-1)-\varphi) c(i-J+(p-1), p-1)\}, \quad i > 0
\]

In fact, by hypothesis and recurrence Eq. (3), we have

\[
s(i, J) = \varphi(i, J) s(i-1, J-1) + \psi(i, J) s(i-p, J-p) \]

\[
= \varphi(i, J) \sum_{t=0}^{2(p-1)} \{\varphi(i-1-(J-1)+t, t) F(i-1, J-1, J-1-t, J-1-t-\varphi) \}
\]

\[
+ c(i-1-(J-1)+t-p, t-p) \}
\]

\[
+ F(i-p, J-p; J-p-t, J-p-t-\varphi) c(i-p-(J-p)+t-p, J-p-t-p) \}
\]

\[
+ F(i-p, J-p; J-p-(p-1), J-p-(p-1)-\varphi) c(i-p-(J-p)+(p-1), J-p-(p-1)-p-1) \}
\]

\[
= \sum_{t=0}^{2(p-1)} \{\varphi(i-J+t, t) \varphi(i, J) F(i-1, J-1, J-1-t, J-1-t-\varphi) \}
\]

\[
+ F(i-p, J-p; J-p-t, J-p-t-\varphi) c(i-p-(J-p)+t-p, J-p-t-p) \}
\]

\[
+ F(i-p, J-p; J-p-(p-1), J-p-(p-1)-\varphi) c(i-p-(J-p)+(p-1), J-p-(p-1)-p-1) \}
\]

\[
= \sum_{t=0}^{2(p-1)} \{\varphi(i-J+t, t) \varphi(i, J) F(i-1, J-1, J-1-t, J-1-t-\varphi) \}
\]

\[
+ F(i-p, J-p; J-p-t, J-p-t-\varphi) c(i-p-(J-p)+t-p, J-p-t-p) \}
\]

\[
+ F(i-p, J-p; J-p-(p-1), J-p-(p-1)-\varphi) c(i-p-(J-p)+(p-1), J-p-(p-1)-p-1), \quad i > 0
\]

In the following we will prove the coefficient of \(c(i-J+p-1, p-1)\) in Eq. (6) and (7) has the equal value (similar to that deduction that prove the coefficient of \(c(i-J+t-p, t-p)\) (\(t = p, p+1, \ldots, 2(p-1)\)) of equal value), that is

\[
\varphi(i, J) F(i-1, J-1, J-1-t, J-1-t-\varphi)
\]

The Wuhan University Journal of Natural Sciences Vol. 10 No. 5 2005

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