Summary. — A fouling transformation is an extension to phase space of the identity transformation on configuration space such that the canonical formalism is preserved. Under such a transformation the new Hamiltonian is no longer the energy. We demonstrate the existence of nontrivial linear fouling transformations for the isotropic two-dimensional oscillator, and study the transformation properties of the resulting Hamiltonian and Lagrangian functions.

1. — Introduction.

In most simple classical mechanical systems, one dynamical variable, namely the energy, is of special significance. This is in part because in the usual treatment the energy becomes the Hamiltonian function over phase space, the function which in the canonical formalism is the infinitesimal generator of translations in time. In this sense, the energy generates the time development of the system. Moreover, when any time-independent canonical transformation is performed on phase space, the energy remains the Hamiltonian. Thus it is generally believed that the energy is the unique dynamical variable which can be used in the canonical formalism (in transformation theory) to generate the motion in time. In this paper we show that this belief, if analyzed more thoroughly than in most studies of classical mechanics, must be seriously modified. Moreover, as will be pointed out in a second paper, this modification raises interesting questions with regard to the transition to quantum mechanics.
A common characterization of a canonical transformation is that it preserves the canonical formalism. That is, a time-independent transformation on phase space from the $q$'s and $p$'s to $Q$'s and $P$'s is canonical if and only if, for every arbitrary Hamiltonian $h(q, p)$, there exists a new Hamiltonian $K(Q, P)$ such that the motion is still generated by Hamilton's canonical equations

$$\dot{Q}_a = \frac{\partial K}{\partial P_a}, \quad \dot{P}_a = -\frac{\partial K}{\partial Q_a};$$

Another common characterization of canonaical is that they preserve the Poisson brackets, i.e. that for any two dynamical variables $F(Q, P) = f(q, p)$ and $G(Q, P) = g(q, p)$ we have

$$\{F, G\} = \frac{\partial F}{\partial Q_a} \frac{\partial G}{\partial P_a} \frac{\partial F}{\partial P_a} \frac{\partial G}{\partial Q_a},$$

and similarly for the Poisson bracket with respect to the $q_a$ and $p_a$. These two characterizations are not equivalent. For instance, the first characterization allows the scaling transformation $Q_a = \lambda q_a, \quad P_a = \lambda p_a$, where $\lambda$ is a constant, while the second does not. It can be shown that from the first characterization, which we adopt in this paper, it follows that Poisson brackets are preserved up to a constant. That is, a transformation is canonical if and only if there exists a constant $\lambda$ such that for every pair of dynamical variables $F$ and $G$

$$\{F, G\} = \lambda \{f, g\}.$$

If $\lambda = 1$, we call the transformation restricted canonical.

It has been pointed out, however, that more general transformations on phase space are also of interest, namely what have been called the canonoids (not quite canonical), which preserve the canonical formalism not for all Hamiltonians, but for only some Hamiltonians or perhaps only one. Thus the canonicals form a subclass of the canonoids. Another subclass of the canonoids is formed by the fouling transformations, for which $Q_a = q_a$. The intersection of these two subclasses consists of the gauge transformations.

(7) Strictly speaking, the gauge transformations are the intersection of the fouling transformations with the restricted canonicals. With $\lambda = 1$, the condition for fouling and the Poisson bracket condition lead to $Q_a = q_a, \quad P_a = p_a + \frac{\partial \Psi}{\partial q_a}$, where $\Psi$ is a function of the $q$'s and of $t$. These we call the gauge transformations.