How to Stay Away from Each Other in a Spherical Universe

1. Tammes' Problem

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Mathematics is full of innocent looking problems which, when pursued, soon grow to majestic proportions and begin to impinge upon the frontiers of research. One such problem is the subject of this two-part article – the problem of packing spherical caps on the surface of a sphere.

Twelve Men on a Sphere

For most practical purposes, we humans are inhabitants of a (two dimensional) spherical universe: we are constrained to eke out our lives on the surface of the earth. Suppose twelve denizens of this spherical universe hate each other so much that they decide to build their houses as far away from each other as possible. We assume that, except for this morbid hatred, these guys have no other constraints: they may build their houses anywhere on the earth’s surface (must be stinkingly rich to have such freedom!). The question is: how should these twelve houses be positioned? The answer depends, of course, on what we mean by the phrase ‘as far away from each other as possible’. Let us take it to mean that the minimum of the pair-wise distances between the twelve houses is to be maximised subjected to the only constraint that they be situated on the spherical surface. With this understanding, it turns out that the problem has a unique solution: the houses must be built at the vertices of an icosahedron inscribed in the sphere. (This is proved in Part 2 of this article.) Now, the icosahedron is a highly structured and symmetrical figure (see Figure 1). Its group of rotational symmetries has order 60 - it is the smallest non-abelian ‘simple’ group. Isn’t it amazing that so much structure and symmetry result as the solution of such a simple minded maximisation problem? Before we finish we
Box 1. Symmetry Group

Given a solid body in Euclidean space (of any dimension), its rotational symmetries are those rotations of the ambient space which send the body (as a whole) onto itself. Clearly, applying two such rotations successively one obtains a third rotational symmetry of the body. It follows that all such rotations form a group with composition as the group law. This is called the isometry group of the body. Other names: symmetry group, automorphism group.

Box 2. The Smallest Non-Abelian Simple Group

Recall that a group is called non-abelian if its law is not commutative: i.e., if the product of two elements depends, in general, on the order in which the multiplication is performed. The order of a finite group means the number of elements in the group. A group is called simple if it has no (proper, non-trivial) normal subgroup. Simple groups are to general groups what prime numbers are to arbitrary integers: they are supposed to be the building blocks of general groups. This is why they are so important. Trivialish examples - in fact the only examples - of abelian simple groups are the symmetry groups of regular \( p \)-gons for various primes \( p \). The symmetry group of the icosahedron (as we have defined it here - excluding reflections) is the non-abelian simple group of the smallest possible order. Abstractly, this group may be described as the group of all even permutations of a set of five objects. All finite simple groups are known. With 26 exceptions (the so called sporadic simple groups) they fall in neat infinite families.

shall see many more examples of this sort of phenomenon. If, however, we wished to build thirteen houses with the above constraint, then nobody knows what is the optimal solution or whether the solution is unique or not!

Pollen Grain

This problem (with an arbitrary number \( n \) in place of twelve) was first raised by the botanist P M L Tammes in 1930. He wanted to explain the observed distribution of the pores on a grain of pollen. He proposed that, for the sake of maximum biological efficiency, the \( n \) pores on the surface of the grain (which may be taken to be a sphere) are placed as far away from each other as possible. Of course, to test this theory, Tammes proposed that for the sake of maximum biological efficiency the pores on a grain of pollen should be placed as far away from each other as possible.