ON TRUNCATION ERROR BOUND FOR MULTIDIMENSIONAL SAMPLING EXPANSION LAPLACE TRANSFORM

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Abstract

The truncation error associated with a given sampling representation is defined as the difference between the signal and an approximating sum utilizing a finite number of terms. In this paper we give uniform bound for truncation error of bandlimited functions in the n dimensional Lebesgue space \( L_p(\mathbb{R}^n) \) associated with multidimensional Shannon sampling representation.

Key words truncation error, band limited function, sampling theorem

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1 Introduction

The well known Whittaker-Kotelnikov-Shannon sampling theorem provide the justification for the use of sampled-date in communication, control and data processing \([1-3]\). This theorem states that every signal function which is bandlimited to \([-\sigma, \sigma]\) can be completely reconstructed from its sampled values \(f(k\pi/\sigma)\). In this case the representation of \(f\) is given by\([3]\)

\[
f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \sin \sigma(t - k\pi/\sigma), \quad \sigma > 0,
\]

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where \( \sin ct = t^{-1} \sin t \) for \( t \neq 0 \), and 1 for \( t = 0 \).

Let \( B_{\sigma,p}(\mathbb{R}) \) be the set of all functions from \( L_p(\mathbb{R}) \) which can be extended to entire functions of exponential type \( \sigma \). According to Schwartz's theorem \(^4\)

\[
B_{\sigma,p}(\mathbb{R}) = \{ f \in L_p(\mathbb{R}) : \text{supp} \hat{f} \subset [-\sigma, \sigma] \},
\]

the \( \hat{f} \) in (2) is the Fourier transform of \( f \) in the sense of generalized functions. In view of the Schwartz theorem, if a function \( f \in L_p(\mathbb{R}) \) is bandlimited to \([-\sigma, \sigma]\), then \( f \in B_{\sigma,2}(\mathbb{R}) \). The result (1) from Shannon is given in the case \( f \in B_{\sigma,2}(\mathbb{R}) \). This result is generalized to the case of \( f \in B_{\sigma,p}(\mathbb{R}), 1 < p < \infty \)\(^5,6\), and also apply to signals \( f(t_1, t_2, \ldots, t_n) \) bandlimited in \( n \)-dimensions\(^7\).

Let \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n_+ \), \( E_v(\mathbb{R}^n) \) denote the class of all entire functions of exponential type \( v \), \( B_v(\mathbb{R}^n) \) be the subset of all functions of \( E_v(\mathbb{R}^n) \) which are bounded on \( \mathbb{R}^n \), and

\[
B_{v,p}(\mathbb{R}^n) := B_v(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)), \quad 1 \leq p \leq \infty, \quad B_{v,\infty}(\mathbb{R}^n)) := B_v(\mathbb{R}^n)).
\]

The following Theorem A gives the multidimensional Shannon sampling representation for \( f \in B_{v,p}(\mathbb{R}^n) \).

**Theorem A\(^7\).** Let \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n_+ \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( f \in B_{v,p}(\mathbb{R}^n), 1 < p < \infty, \) then

(a)

\[
f(x) = \sum_{k \in \mathbb{Z}^n} f(k\pi/v) \sin_n(v(x - k\pi/v)),
\]

and the series on right converges uniformly on \( \mathbb{R}^n \).

(b)

\[
\left\| f - \sum_{|k| \leq n_l} f(k\pi/v) \sin_n(v(x - k\pi/v)) \right\|_{p(\mathbb{R}^n)} \to 0, \quad n_l \to \infty, \quad l = 1, 2, \ldots, n,
\]

where \( k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n, k\pi/v = \{k_1/v_1, k_2/v_2, \ldots, k_n/v_n \}; \sin_n x = \prod_{j=1}^n \sin x_j, \sin cx = x^{-1} \sin x, \) if \( x \neq 0 \), and 1 if \( x = 0 \).

For \( f \in L_p(\mathbb{R}^n) \cap C(\mathbb{R}^n), 1 \leq p \leq \infty, N = (N_1, \ldots, N_n) \in \mathbb{N}^n \), let

\[
E_N f(x) = \sum_{|k_1| \leq N_1} \cdots \sum_{|k_2| \leq N_2} \sum_{|k_1| \leq N_1} f(k\pi/v) \sin_n(v(x - k\pi/v))
\]

and the truncation error is defined as \( f(x) - E_N f(x) \).

Shannon’s expansion requires us to know the exact values of \( f \) at infinitely many points and to sum an infinite series, but in practical situations, only finitely many samples are available so we would like to develop bounds on the size of the truncation error.