CONSTRUCTION OF SOME KIESSWETTER-LIKE FUNCTIONS - THE CONTINUOUS BUT NON-DIFFERENTIABLE FUNCTION DEFINED BY QUINARY DECIMAL

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Abstract

In this paper, we construct some continuous but non-differentiable functions defined by quinary decimal, that are Kiesswetter-like functions. We discuss their properties, then investigate the Hausdorff dimensions of graphs of these functions and give a detailed proof.

Key words Kiesswetter-like functions, continuous but non-differentiable, quinary decimal, iterated function system, inequality; Hausdorff dimension

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1 Introduction

In 1966, Karl Kiesswetter\(^1\) gave a simple example of everywhere continuous and nowhere differentiable function, namely, so-called Kiesswetter's function.

Let \(x \in [0,1)\), its quartenary expansion:

\[
x = \sum_{n=1}^{\infty} \frac{x_k}{4^k}, \quad x_k \in \{0, 1, 2, 3\}.
\]

Then, Kiesswetter's function is defined as follows:

\[
K(x) = \sum_{n=1}^{\infty} \frac{(-1)^n U(x_n)}{2^n},
\]
where

\[
U(x_n) = \begin{cases} 
    x_n - 2, & x_n = 1, 2, 3, \\
    0, & x_n = 0,
\end{cases}
\]

and \(N_n\) is the number of \(x_k = 0\) only if \(k < n\).

This is a very noticeable fractal function (see [2],[3]). G.A.Edgar regarded it as one of the nineteen typical fractal articles that included Weierstrass's function, Von Koch curve and Cantor set etc, and introduced them in the collected papers: "Classics on Fractal"[4].

Since the continuous but non-differentiable function was proposed for the first time by Weierstrass in 1872, some other examples have been obtained. A question which naturally arises is how to construct the class of functions, and what is the relationship between classical fractals and modern fractals?

Recently, a general method by using the combining \(b\)-adic expansion with iterated function system to define the everywhere continuous and nowhere differentiable functions is found in papers [5] and [6]. However, their discussion is merely on the situation: \(f(x) : [0, 1] \rightarrow [0, 1]\), while Kiesswetter's function showed that \(K(x) : [0, 1] \rightarrow [-1, 1]\). On the base of [5] and [6], we extend the Kiesswetter's function on the situation of quinary decimal and analyse their properties in this paper. Finally, we have proved that the Hausdorff dimensions of graphs of these functions are equal to \(2 - \frac{\log 3}{\log 5}\).

## 2 Construction of some Kiesswetter-like functions

Now, we construct some continuous but non-differentiable function defined by quinary decimal, which are similar to Kiesswetter's functions.

With the same idea of [5] and [6], consider the following five affine-mapping expressions \(W_j(j = 0, 1, 2, 3, 4)\):

\[
W_j = \begin{pmatrix}
\frac{1}{5} & 0 \\
0 & (-1)^{\alpha_j} \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix}
+ \begin{pmatrix}
\frac{j}{5} \\
\frac{U(j)}{3}
\end{pmatrix}, \\
\begin{cases} 
    x = \frac{j}{5} + \frac{\xi}{5}, \\
    f(x) = (-1)^{\alpha_j} \frac{1}{3} f(\xi) + \frac{U(j)}{3},
\end{cases}
\]

\(j = 0, 1, 2, 3, 4, \quad (1)\)

where \(U(j)\) are some constants depending on \(j\) and \(\alpha_j = 0\) or 1. Let \(\eta = f(\xi)\) and \(y = f(x)\), then (1) can be written as

\[
\begin{cases} 
    x = \frac{j}{5} + \frac{\xi}{5}, \\
    f(x) = (-1)^{\alpha_j} \frac{1}{3} f(\xi) + \frac{U(j)}{3},
\end{cases}
\quad j = 0, 1, 2, 3, 4.
\]

(2)

Suppose that \(x \in [0, 1]\), its quinary decimal expansion is

\[
x = \sum_{k=1}^{\infty} \frac{x_k}{5^k}, \quad x_k \in \{0, 1, 2, 3, 4\}.
\]