THE PACKING MEASURE OF A CLASS
OF GENERALIZED SIERPINSKI CARPET

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Received Feb. 5, 2004

Abstract
For \( \frac{1}{4} < a < \frac{\sqrt{2}}{4} \), let \( S_1(x) = ax, S_2(x) = 1 - a + ax, x \in [0,1] \). \( C_a \) is the attractor of the iterated
function system \( \{S_1, S_2\} \), then the packing measure of \( C_a \times C_a \) is

\[
P^{\alpha}(C_a \times C_a) = 4 \cdot 2^{\alpha}(1 - a)^{\alpha},
\]

where \( \alpha(a) = -\log_4 4 \).

Key words  self-similar, packing dimension and measure, generalized sierpinski carpet

AMS(2000)subject classification  28A78, 28A80

1 Introduction and Main Theorem

Computing and estimating the dimension and measure of fractal sets are one of the import-
tant problems in fractal geometry. Generally speaking, computing the Hausdorff and packing
dimension, especially the Hausdorff and packing measure are very difficult. For instance, in [8],
Sullivan D. points out: "In the study of geometric limit set sometimes it is the Hausdorff mea-
ure which is important and sometimes the packing measure, a theme which continue to appear
in dynamics. Although the importance of packing measure has been becoming more and more

* This project was supported in part by the Foundations of the Natural Science Committee,
Guangdong Province and Zhongshan University Advanced Research Centre, China.
apparent, they remain somewhat more difficult to deal with” in [6], we get the exact value of packing measure of $C \times C$ with packing dimension more than 1, where $C \times C$ is the Cartesian product of Cantor set. Reference [7] obtained the exact value of packing measure of $C_a \times C_a$, where $C_a$ is the attractor of the iterated function system \{S_1, S_2\}, $S_1(x) = ax$, $S_2(x) = 1-a+ax$, $x \in [0,1]$ and $0 < a \leq \frac{1}{4}$. In this paper, when $\frac{1}{4} < a < \frac{\sqrt{2}}{4}$, the exact value of the packing measure of $C_a \times C_a$ is obtained.

Let $0 < a < \frac{1}{2}$. Take $E_0$ to be the unit square in $R^2$ and delete all but the four corner squares (including their boundary) of side length $a$ to obtain $E_1$. Inductively, for $n \geq 1$ continue in this way, at the $n$-th stage replacing each square of $E_{n-1}$ by its four corner squares of side length $a^n$ to get $E_n$. We obtain $E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots$. The non-empty set $C_a \times C_a = \bigcap_{n=0}^{\infty} E_n$ is called the Generalized Sierpinski Carpet with contracting ratio $a$. The packing dimension of $C_a \times C_a$ is (see [1])

$$s = s(a) = \dim_P(C_a \times C_a) = -\log_a 4.$$

Take the origin to be a vertex of $E_0$ (see Figure 1). Then $E_0 = [0,1] \times [0,1]$ and $C_a \times C_a$ can be regarded as the attractor of the iterated function system \{f_1, f_2, f_3, f_4\} with the strong separation condition, where $f_i(x) = ax + b_i$, $i = 1, 2, 3, 4$, where $b_1 = (0,0)$, $b_2 = (1-a,0)$, $b_3 = (1-a,1-a)$, $b_4 = (0,1-a)$. We have

$$C_a \times C_a = \bigcup_{n=1}^{4} f_n(C_a \times C_a).$$

![Figure 1](image-url)

When $0 < a \leq \frac{1}{4}$, $0 < s \leq 1$, $P^s(C_a \times C_a) = 4 \cdot 2^s(1-a)^s$ (see [7]).

The main result of this paper is the following theorem.

**Theorem.** Let $\frac{1}{4} < a < \frac{\sqrt{2}}{4}$, then the packing measure of $C_a \times C_a$ is

$$P^s(C_a \times C_a) = 4 \cdot 2^s(1-a)^s,$$

where $s = -\log_a 4$. 