1. Introduction

In this paper we study uniform approximation by harmonic functions on arbitrary closed subsets of a Riemannian manifold. Thus our goal is to extend a number of results obtained in earlier papers concerning uniform harmonic approximation on unbounded closed subsets of Riemann surfaces or regions in Euclidean space $\mathbb{R}^n$. In this setting we obtain a localization theorem, and extensions of theorems of Runge, Keldysh, and Debiard and Gaveau.

The original work of Runge was a study of holomorphic approximation in the plane, and theorems of this type are now known for solutions of elliptic partial differential equations. These results can be stated in very general forms which include, as special cases, approximation theorems both for holomorphic functions and for harmonic functions; results of this type include the Runge theorem for compact subsets of manifolds due to Lax and Malgrange, as presented in [Nar, 3.10.7], and the Runge theorem for closed subsets of $\mathbb{R}^n$ given by Dufresnoy, Gauthier, and Ow [DGO, Theorem 2]. By contrast, our harmonic Runge theorems 5.1, 9.2 and 9.3 would all be false for holomorphic functions on Riemann surfaces ([GH1, Example 2], [B, Introduction]).

We begin in Section 2 with an elementary discussion of the notions concerning Riemannian manifolds which are needed to state global versions of harmonic approximation theorems; in order to make the paper more accessible to readers whose main interests are not in Riemannian geometry, we have included definitions of several basic concepts. In Section 3 we study harmonic functions on a Riemannian manifold. In Section 4 we discuss harmonic classification theory for Riemannian manifolds, and construct the kernel functions which are basic to our approach: Green functions for hyperbolic manifolds and Evans functions on open parabolic manifolds. We then give a Runge theorem for approximation by harmonic functions with singularities in Section 5. We study in Section 6 a localization operator analogous to the one used by Vitushkin in the study of holomorphic approximation in the complex plane, and we use this operator in Section 7 to prove our localization theorem for harmonic approximation. In Section 8 we combine our

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earlier work with results from axiomatic potential theory to obtain our extensions of the theorems of Keldysh and Debiard and Gaveau. In Section 9 we give Runge theorems for approximation by harmonic functions with prescribed singularities, and approximation by global harmonic functions; and as an application of our approximation theorems we prove a Mittag-Leffler theorem for harmonic functions on open manifolds.

It is a pleasure to acknowledge many helpful conversations with P. M. Gauthier concerning this paper. In particular, the use of axiomatic potential theory to prove our extension of the Keldysh theorem was suggested by him, and the Mittag-Leffler theorem in Section 9 was obtained jointly with him.

Note added January 1993. Several new results on approximation by global harmonic functions have been obtained in recent papers by Bagby, Gardiner, Gauthier, Goldstein, and GowriSankaran.

2. Riemannian manifolds

Throughout this paper we let $\Omega$ denote a (connected, class $C^\infty$) manifold. If $V$ is a coordinate region in $\Omega$ with coordinates $x = (x^1, ..., x^n)$, and $\tilde{V}$ is an overlapping coordinate region with coordinates $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^n)$, we then obtain a $C^\infty$ map $x \rightarrow \tilde{x}$ whose Jacobian matrix $\partial \tilde{x}/\partial x$ has $ij$th entry equal to $\partial \tilde{x}^i/\partial x^j$. We assume that $\Omega$ is orientable, which means that we may select a family of coordinate regions covering $\Omega$ such that for every pair of overlapping coordinate regions $V$ and $\tilde{V}$ from the family we have $\det(\partial \tilde{x}/\partial x) > 0$; for the rest of this paper we suppose a maximal family of coordinate regions with this property is fixed, and all coordinate regions are understood to be from this family. If $A$ is any subset of $\Omega$, then $C(A)$ denotes the set of all continuous real-valued functions on $A$. If $V \subset \Omega$ is open, then functions in $C^\infty(V)$ are called smooth; we let $C_0(V)$ [resp. $C_0^\infty(V)$] denote the set of all functions $u \in C(V)$ [resp. $u \in C^\infty(V)$] which vanish off a compact subset of $V$. If a local coordinate mapping carries a subregion of $\Omega$ onto a subregion of $\mathbb{R}^n$ containing a closed ball $\{x \in \mathbb{R}^n : |x - x_0| \leq r\}$, $r > 0$, then the preimage of the open ball $\{x \in \mathbb{R}^n : |x - x_0| < r\}$ is called a parametric ball in $\Omega$. If $u \in C^\infty(\Omega)$, then for each coordinate region $V$ with coordinates $(x^1, ..., x^n)$ we may compute the partial derivatives $a_i = \partial u/\partial x^i$; notice that the rule which assigns to each $V$ the function $a = (a_1, ..., a_n) : V \rightarrow \mathbb{R}^n$ is a covariant 1-tensor (or 1-form) on $\Omega$, which means that for overlapping coordinate regions $V$ and $\tilde{V}$ we have the matrix equation $\tilde{a} \equiv a \partial x/\partial \tilde{x}$. A smooth contravariant 1-tensor (or smooth vector field) on $\Omega$ is a function $X$ assigning to each point $p \in \Omega$ an element of the tangent space $T_p(\Omega)$ at $p$, which is smooth in the sense that in each coordinate region $V$,