THE POINCARÉ SERIES AND THE CONFORMAL MEASURE OF CONICAL AND MYRBERG LIMIT POINTS

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1. Introduction

The subject this paper is the relationship of three notions pertaining to a discrete group $G$ of Möbius transformations acting on the closed euclidean $(n + 1)$-ball $B^{n+1} \subset \mathbb{R}^{n+1}$. The first notion is that of a conformal $G$-measure. Let $A \subset B^{n+1}$ be $G$-invariant. A measure on $A$ is such a measure of dimension $\delta$ if the set of measurable sets includes all Borel sets, if it is finite and if it satisfies the transformation rule

$$\mu(gX) = \int_X |g'|^\delta d\mu$$

for all $g \in G$ and all measurable $X$; here $|g'|$ is the norm of the differential of $g$. Conformal measures were considered by Sullivan in [S2] who extended a method of Patterson [P] to construct a non-trivial conformal measure supported by the limit set $L(G)$ of $G$. More familiar examples are the Hausdorff $(n + 1)$-measure on $B^{n+1}$ and the Hausdorff $n$-measure on $S^n$.

The Poincaré series of exponent $\delta$ can be defined as the series

$$\sum_{g \in G} e^{-\delta d(z, g(a))}$$

where $z$ and $a$ are in the open ball $B^{n+1}$ and $d$ is the hyperbolic metric of $B^{n+1}$ (cf. [S2]). The convergence or divergence of the series does not depend on $z$ and $a$ and there is a critical value $\delta_G$, called the exponent of convergence of $G$, such that the Poincaré series converges for $\delta > \delta_G$ and diverges for $\delta < \delta_G$; for $\delta = \delta_G$ the series may converge or diverge. The dimension of the Patterson–Sullivan measure mentioned above is $\delta_G$.

The third principal notion which we need is the conical limit set $\Lambda(G)$ of $G$ which is the set of points $x$ in the unit $n$-sphere $S^n$ for which there is a sequence $g_i \in G$ such that if $z \in B^{n+1}$ and $L$ is any hyperbolic line with endpoint $x$, then $g_i(z) \to x$ as $i \to \infty$ and the points $g_i(z)$ are of bounded hyperbolic distance from $L$.

If $\mu$ is the Hausdorff $n$-measure of the $n$-sphere $S^n$, then it is known that the Poincaré series diverges at the exponent $\delta = n$ if and only if the conical limit points
have full $\mu$-measure in $S^n$. For $n = 1$ this result was due to Hopf. Later this has been generalized for all $n$ by Sullivan [S1] and Thurston; Thurston’s proof can be found in [AH], [AG], or [NI].

It is easy to show that if the Poincaré series of exponent $\delta$ converges, then the conical limit point set has zero $\mu$-measure for any conformal measure $\mu$ of dimension $\delta$ ([S2, Corollary 20], [NI, Theorem 4.4.1]). Conversely, Nicholls has proved that if $\mu$ is a non-trivial conformal measure of dimension $\delta$ on the limit set, which gives zero measure for conical limit points, then the Poincaré series converges at the exponent $\delta$. His approach was akin to Sullivan’s proof for the case where $\mu$ is the Hausdorff $n$-measure and $\delta = n$. We will prove this same result for general conformal measures (Theorem 3A) with ideas not far from the ones in Thurston’s proof which has the advantage of a more direct approach.

In Section 4 we consider the class of conical limit points called Myrberg points and $x \in L(G)$ is such a point if, given $z \in B^{n+1}$, any hyperbolic line $L$ with endpoints in $L(G)$ can be approximated arbitrarily closely by the $G$-transforms of the hyperbolic ray with endpoints $z$ and $x$. This means that given neighbourhoods $U$ and $V$ of the endpoints of $L$, then some $g \in G$ maps $z$ to $U$ and $x$ to $V$. We denote Myrberg points by $M(G)$ and this is a subset of the conical limit point set (cf. [BM, Theorem 1]).

Myrberg [M] showed that $M(G)$ has full linear measure in $S^1$ if $G$ is a Fuchsian group of finite hyperbolic volume. Agard [AG] has generalized this and shown that the $n$-dimensional Hausdorff measure gives full measure for $M(G)$ in $S^n$ and Nakanishi [NA] has proved the same result for geometrically finite Kleinian groups of the first kind. Our contribution (Theorem 4A) is to show that the measure of of conical limit points which are not Myrberg points vanishes for any conformal measure.

Finally, in Section 5, we will give a slightly weaker characterization of Myrberg points according to which $x$ is such a point if and only if whenever $L$ is a hyperbolic line or ray with one endpoint $x$, then the $G$-transforms of $L$ are dense in the set of hyperbolic lines whose endpoints are in $L(G)$. Note that here we consider hyperbolic lines to be unoriented whereas in the definition of Myrberg points we approximated by oriented hyperbolic lines.

**Conformal measures and the half-space model.** We have given these definition using the $(n + 1)$-ball $B^{n+1}$ as the model for the $(n + 1)$-dimensional hyperbolic space. The main advantage is that the finiteness condition for conformal measures is easily expressed in this situation. However, one of our basic methods is to rescale the situation in the euclidean metric and this is easier if we use the half-space model $H^{n+1} = R^n \times (0, \infty)$ for the hyperbolic space. We denote by $H^{n+1}_\partial = H^{n+1} \cup R^n \cup \{\infty\}$ the hyperbolic $(n + 1)$-space with boundary.

If we have a Möbius group $G$ on $B^{n+1}$, we can always transfer it to $H^{n+1}_\partial$ by