THE BEST APPROXIMATION CONSTANT FOR
THE JACKSON'S TYPE OPERATOR $J_{n,3}(f; x)$

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Abstract

In this paper we obtain the best approximation constant of function $f(x) \in C_{2\pi}$ by the
Jackson's type operator $J_{n,3}(f; x)$, i.e.

$$\|J_{n,3}(f; x) - f(x)\| \leq \left( 4 - \frac{6}{\pi} \right) \omega(f; \frac{1}{n}),$$

$$\|J_{n,3}(f; x) - f(x)\| \leq \left( 8 - \frac{17}{\pi} \right) \omega_2(f; \frac{1}{n}).$$

Let function $f(x)$ be an integrable function with period $2\pi$, the Jackson's type opera-
tor is defined by

$$J_{n,p}(f; x) = \frac{1}{I_{n,p}} \int_{-\pi}^{\pi} f(t) \left( \frac{\sin \frac{n(t-x)}{2}}{\sin \frac{t-x}{2}} \right)^{2p} dt$$

(1)

where $n$ and $p$ are natural numbers, i.e., $n, p \in \mathbb{N}$, and

$$I_{n,p} = \int_{-\pi}^{\pi} \left( \frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^{2p} dt.$$  

(2)

In special, $J_{n,1}(f; x)$ is the Fejér operator; $J_{n,2}(f; x)$ is the Jackson operator.

Let $C_{2\pi}$ denote the class of continuous functions with period $2\pi$, and let $\omega(f; \delta)$ de-
ote a modulus of continuity of a function $f(x)$, let $\omega_2(f; \frac{1}{n})$ denote a continuity
modulus of second degree of $f(x)$, i.e.

$$\omega_2(f; \delta) = \max_{0 \leq h \leq \delta} |f(x+h) + f(x-h) - 2f(x)|.$$
In this paper, we obtain the best approximation constant of \( f(x) \in C_{2x} \) by the Jackson’s type operator \( J_{n,3}(f,x) \).

By means of the polynomial theorem and Euler’s formula, we can show the following.

**Lemma.** Let \( n \in \mathbb{N} \), the equality
\[
\left( \frac{\sin nt}{\sin t} \right)^6 = \frac{n(11n^4 + 5n^2 + 4)}{20} + 2 \sum_{k=1}^{3n-3} a_k^{(n)} \cos 2kt
\]
holds, where
\[
a_k^{(n)} = \begin{cases} \binom{3n+2-k}{5} - 6 \binom{2n+2-k}{5} + 15 \binom{n+2-k}{5}, & 1 \leq k \leq n-3, \\
\binom{3n+2-k}{5} - 6 \binom{2n+2-k}{5}, & n-2 \leq k \leq 2n-3, \\
\binom{3n+2-k}{5}, & 2n-2 \leq k \leq 3n-3. \\
\end{cases}
\]

**Corollary 1.**
\[
\int_{-\pi}^{\pi} \left( \frac{\sin nt}{\sin t} \right)^6 dt = \frac{n(11n^4 + 5n^2 + 4)\pi}{10}.
\]

**Theorem 1.** Suppose \( f(x) \in C_{2x} \) and \( n \in \mathbb{N} \), the Jacksons type operator is defined as follows
\[
J_{n,3}(f; x) = \frac{1}{I_{n,3}} \int_{-\pi}^{\pi} f(t) \left( \frac{\sin\frac{n(t-x)}{2}}{\sin\frac{t-x}{2}} \right)^6 dt,
\]
where
\[
I_{n,3} = \int_{-\pi}^{\pi} \left( \frac{\sin\frac{n t}{2}}{\sin\frac{t}{2}} \right)^6 dt = \frac{n(11n^4 + 5n^2 + 4)\pi}{10},
\]
then for any \( n \geq 1 \), the inequality
\[
\| J_{n,3}(f; x) - f(x) \|_e \leq \left( 4 - \frac{6}{\pi} \right) \omega\left( f; \frac{1}{n} \right)
\]
holds, and
\[
\sup_{n \in \mathbb{N}} \sup_{f \in C_{2x}, \ f \neq \text{const}} \left\{ \frac{\| J_{n,3}(f; x) - f(x) \|_e}{\omega\left( f; \frac{1}{n} \right)} : f \neq \text{const} \right\} = 4 - \frac{6}{\pi}
\]
the constant \( 4 - \frac{6}{\pi} \) is best possible.