Kaun Banega Crorepati – A Million Dollars for a Mathematician

2. Poincaré Conjecture

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I will now embark on explaining as best as I can to the non-mathematician what the Poincaré conjecture is all about.

Poincaré Conjecture

The Poincaré conjecture is a problem in topology, an area which, as I mentioned earlier, is essentially the creation of Poincaré. The topologist studies geometric objects looking for properties that remain unchanged when the object is moved, stretched, contracted, bent – when it is subjected to a very wide class of ‘transformations’ called ‘topological transformations’.

Before I explain what a topological transformation is, I must first say what a geometric object is for a topologist. Familiar objects such as triangles, polyhedra, circles, cubes and spheres which figure in Euclidean geometry are of course among geometric objects for topologists; one goes much farther: any subset, any aggregate of points of any shape or size in 3-dimensional space is a geometric object. But topology does not stop even there; it takes in the study of subsets of Euclidean spaces of all possible dimensions. The n-dimensional (Euclidean) space \( \mathbb{R}^n \) is the collection of all possible \( n \)-tuples \((x_1, ..., x_n)\) of real numbers: just as a point in our familiar 3-dimensional space is a triple \((x_1, x_2, x_3)\) of real numbers (the Cartesian coordinates of the point), a point in \( n \)-dimensional Euclidean space is an \( n \)-tuple. A geometric object in topology is any arbitrary subset of some \( n \)-dimensional Euclidean space.

We do not have a visual picture of geometric objects in higher (\( > 3 \)) dimensional Euclidean spaces, but they do confront us in the physical world. An event for the physicist takes place at a
point in 3-dimensional space at a certain time. To specify the event then we need 4 numbers, three to indicate the point in $\mathbb{R}^3$ and the fourth to give the time of the event. Thus we encounter 4-dimensional (Euclidean) space – the space-time continuum in physics.

Another example in physics arises in the study of motion of rigid bodies. If we fix four points $O, P, Q$ and $R$ on the rigid body such that $OP, OQ$ and $OR$ are mutually perpendicular, then the position of the rigid body in space is completely determined if one knows the coordinates of all these four points in 3-dimensional space and these make up four triples of numbers or equivalently twelve numbers.

Thus every possible position that a rigid body occupies in space (one calls this the configuration space) is determined by twelve numbers and twelve numbers give a point in $\mathbb{R}^{12}$, the Euclidean space of dimension twelve; in other words each position of a rigid body in 3-space determines a point in $\mathbb{R}^{12}$. But not every point in $\mathbb{R}^{12}$ will correspond to one such position – the four triples cannot be chosen arbitrarily as the distances between any two of $O, P, Q$ and $R$ is fixed. In other words the configuration space is a subset of $\mathbb{R}^{12}$ but not all of it. Well, all this goes to show that there are reasons other than the mathematician's curiosity and imagination to study geometric objects in Euclidean spaces of higher dimensions.

Continuity is the basic concept on which topology rests and I