Let $R$ be a normed linear space, $K$ be an arbitrary convex subset of an $n$-dimensional subspace $\Phi_n \subset R$. This paper first gives a general characterization for a best approximation from $K$ in form of "zero in the convex hull". Applying it to the uniform approximation by generalized polynomials with restricted ranges, we get further an alternation characterization. Our results contains the special cases of interpolatory approximation, positive approximation, copositive approximation, and the classical characterizations in forms of convex hull and alternation in approximation without restriction.

Owing to the diversity of approximation means and the functions approximated, the problem of characterization of a best approximation is still an open question in many cases, which has been investigated by many authors for a long time.

Let $R$ be a normed linear space, and $\{\varphi_j\}_{j=1}^n \subset R$ be linearly independent. We call $\Phi_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}$ the set of generalized polynomials. This paper first gives a general characterization for a best approximation from an arbitrary convex subset $K$ of $\Phi_n$ (especially, $K$ can be a linear subspace) to a given function $f \in R \setminus K$. This result is applicable extensively. Applying it to the uniform approximation by generalized polynomials with restricted ranges, we can get the characterizations in forms of convex hull and alternation, which contains and generalizes the results of [1]–[3] and the classical characterization theorems in approximation without restriction.

\section{Main Results}

In general, given $p \in K$ we can directly discuss the characterization of $p$ to be a best approximation to $f$ from $K$ (even if the existence of the best approximation is open to question). In doing this, we first define the inner product in $\Phi_n$ as
(q_1, q_2) := \sum_{i=1}^{n} a_i^{(1)} a_i^{(2)} \quad (q_i = \sum_{i=1}^{n} a_i^{(i)} \varphi_i, \quad i = 1, 2).

(In general, the norm derived by the inner product is not necessarily the norm \| \cdot \| of space \( R \).) Let

\[ K_p := \{ q \in \Phi : \| f - q \| \leq \| f - p \| \}, \]

and

\[ K^* := h\{ q \in \Phi : g \neq 0; (g, q - p) \leq 0, \forall q \in K \}, \]

\[ K^*_p := h\{ q \in \Phi : g \neq 0; (g, q - p) \leq 0, \forall q \in K_p \}, \]

where \( h Q \) (or \( h(Q) \)) denotes the convex hull of the subset \( Q \) of \( \Phi \), that is

\[ h Q = h(Q) := \{ q : q = \sum q_i \varphi_i, \varphi_i \in Q, q_i \geq 0, \sum q_i = 1, m \in \mathbb{N} \}, \]

where \( \mathbb{N} = \{1, 2, 3, \ldots\} \). Then we have the characterization of a best approximation from \( K \) as follows.

**Theorem 1.** Assume that \( \Phi \subset R \), the convex set \( K \subset \Phi \), \( f \in R \setminus K \). If \( p \in K \) is not a best approximation to \( f \) from \( \Phi \), or the best approximation to \( f \) from \( \Phi \) is unique, then \( p \) is a best approximation to \( f \) from \( K \) if and only if, there exists a non-vanishing vector \( g \in K^* \) for which \( 0 \in h(\{ g \} \cup K^*_p) \), which means \( 0 \in h(K^*_p) \) if \( K^* = \emptyset \).

Theorem 1 is applicable extensively because no confinement have been added to the space \( R \) and its norm, and to the convex restriction conditions for \( K \). For directly applying Theorem 1 in various concrete cases, it only need to determine the structure of \( K^* \) and \( K^*_p \) for each case. For this purpose, we give Theorems 2 and 3 in which follows.

First, \( \Phi = \text{span}\{ \varphi_1, \ldots, \varphi_n \} \) is said to be an extended complete Tchebycheff subspace on interval \([a, b]\), if \( \varphi_i \in C^{n+1}[a, b], i = 1, \ldots, n \); for \( r = 1, \ldots, n \) and any given group of points \( a \leq \xi_1 \leq \cdots \leq \xi_r \leq b \), it follows that

\[
\Phi^* \left( \begin{array}{cccc}
1, & 2, & \ldots, & r \\
\xi_1, \xi_2, \ldots, \xi_r
\end{array} \right) := \left| \begin{array}{cccc}
\tilde{\varphi}_1(\xi_1) & \tilde{\varphi}_1(\xi_2) & \cdots & \tilde{\varphi}_1(\xi_r) \\
\tilde{\varphi}_2(\xi_1) & \tilde{\varphi}_2(\xi_2) & \cdots & \tilde{\varphi}_2(\xi_r) \\
& \cdots & & \\
& & & \\
\tilde{\varphi}_r(\xi_1) & \tilde{\varphi}_r(\xi_2) & \cdots & \tilde{\varphi}_r(\xi_r)
\end{array} \right| > 0,
\]

(1)

where for fixed \( j, \tilde{\varphi}_i(\xi_j) = \varphi_i(\xi_j) \) if \( \xi_{i-1} < \xi_j, \tilde{\varphi}_i(\xi_j) = \varphi_i^{(k)}(\xi_j) \) if \( \xi_{i-k-1} < \xi_{i-k} = \cdots = \xi_j, 1 \leq i \leq r \).

The set of generalized polynomials with restricted ranges is defined as

\[ K(l, u) := \{ q \in \Phi : l(x) \leq q(x) \leq u(x), x \in [a, b] \}, \]