PADE APPROXIMANTS AS LIMITS OF RATIONAL FUNCTIONS OF BEST APPROXIMATION IN ORLICZ SPACE*

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Abstract

In this paper, we prove that the best rational approximation of a given analytic function in Orlicz space $L^\Phi(G)$, where $G = \{ |z| \leq \varepsilon \}$, converges to the Padé approximants of the function as the measure of $G$ approaches zero.

1. Introduction

We shall call a rational function being of type $(n, l)$ provided it can be written in the form

$$\frac{s_0 + s_1z + \cdots + s_nz^n}{t_0 + t_1z + \cdots + t_lz^l}, \quad \sum |t_k| \neq 0. \quad (1)$$

The Padé approximants to a given analytic function $f(z)$ is a rational function $P_{nl}(z)$ of type $(n, l)$ with contact of the highest order at the origin to

$$f(z) = a_0 + a_1z + \cdots + a_{n+l}z^{n+l} + O(z^{n+l+1}), \quad a_0 \neq 0. \quad (2)$$

It is shown in [1] that provided a certain determinant consisting of the $a_k$s is not zero, the rational function $R_{nl}(e, z)$ of type $(n, l)$ of the best approximation to $f(z)$ in the Chebyshev sense on the disc $G = \{ |z| \leq \varepsilon \}$, as $\varepsilon \to 0$, approaches the Padé function $P_{nl}(z)$ of $f(z)$ uniformly on any closed set within which $P_{nl}(z)$ is analytic. In this paper we prove the analogous theorem in Orlicz space.

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2. Notations and Results of Orlicz Space

**Definition 1.** A real function \( M(u) \) defined in \( (-\infty, +\infty) \) is called an \( N \)-function if it can be written in the form \( M(u) = \int_0^\infty p(t)dt \), where \( p(t) \) is an increasing right continuous function. and \( p(t) \) is positive if \( t > 0, p(0) = 0, p(\infty) = \lim_{t \to \infty} p(t) = \infty \). Let 
\[
q(s) = \sup_{t \leq s} tN(v) = \int_0^s q(t) \, dt.
\]
we call \( N(v) \) the complementary \( N \)-function of \( M(u) \).

**Definition 2.** Let \( G \) be a closed bounded set in \( \mathbb{R}^n \), let \( M(u), N(v) \) be \( N \)-functions which are complementary each other, let \( u(x) \) be a real value function, we call
\[
\int_G M(u(x)) \, dx
\]
to be a modular of \( u(z) \) respect to \( M \), and denote it by \( \rho(u, M) \), we write
\[
L_M = \{u(x); \rho(u, M) < \infty\}
\]
\[
L_M^* = \{u(x); \exists k > 0, \rho(ku, M) < \infty\}
\]
we define norm of \( u(z) \) in \( L_M^* \) as follows
\[
\|u(x)\|_M = \sup_{\|u, M\|} \left| \int u(z)v(z) \, dz \right|,
\]
\( L_M^* \) is called Orlicz space.

In the investigation of Orlicz space \( L_M^*(G) \), we usually consider the \( N \)-functions which satisfy the following conditions: \( \Delta_2, \Delta_3 \).

**Definition 3** An \( N \)-function \( M(u) \) satisfies condition \( \Delta_2 \) if there exist \( k \geq 2 \) and \( u_0 \geq 0 \), such that when \( u \geq u_0, M(2u) \leq kM(u) \).

**Definition 4** An \( N \)-function \( M(u) \) satisfies \( \Delta_3 \) if there exist \( k > 0 \) and \( u_0 \geq 0 \), such that when \( u \geq u_0, uM(u) \leq M(ku) \).

**Definition 5** An \( N \)-function \( M(u) \) satisfies condition \( \Delta_2^* \) if \( \exists k > 0 \) and \( u_0 \geq 0 \), when \( u \geq u_0, M(u) \leq ku^2 \).

**Definition 6** An \( N \)-function \( M(u) \) satisfies condition \( \Delta_3^* \) if 
\[
\lim_{u \to \infty} \frac{M(u)}{u^2} = \infty.
\]

**Theorem 1** Let \( N_{\Delta_2^*}, N_{\Delta_3^*} \) denote the class of \( N \)-functions which satisfy conditions