EQUIVALENCES BETWEEN BLOCH TYPE SPACES AND BMO SPACES ON THE HYPERBOLIC DOMAINS IN C

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Abstract

For an analytic function f on the hyperbolic domain \( \Omega \) in \( \mathbb{C} \), the following conclusions are obtained:

(i) \( f \in B(\Omega) = BMOA(\Omega,m) \) if and only if \( \text{Re} f \in B(\Omega) = BMOH(\Omega,m) \).

(ii) \( Q^B(\Omega) = BMO^I(\Omega,m) \) if and only if \( C(\Omega) = \inf\{ \lambda_\Omega(z) \cdot \delta_\Omega(z) : z \in \Omega \} > 0 \). Also, some applications to automorphic functions are considered.

1. Introduction

Let \( D = \{ w : |w| < 1 \} \) be the unit disk in the finite complex plane \( \mathbb{C} \) and \( \lambda_D(w) = |d w|/(1 - |w|^2) \) the hyperbolic metric on \( D \). Also, suppose that \( \Omega \) is a hyperbolic domain in \( \mathbb{C} \), that is to say, there exists a Fuchsian group \( \Gamma \) acting on \( D \) such that \( \Omega \) is conformally equivalent to the quotient space \( D/\Gamma \). Denote by \( \lambda_\Omega(z)dz \) the hyperbolic metric on \( \Omega \), where \( \lambda_\Omega(z) \) can be determined by the following equality:

\[
\lambda_\Omega(\pi(w)) \cdot |\pi'(w)| = \lambda_D(w), \quad w \in D
\] (1.1)

here \( \pi \) is an analytic universal covering map of \( D \) onto \( \Omega \). Further for all \( \gamma \in \Gamma \),

\[
\lambda_\Omega(\pi \circ \gamma(w)) \cdot |(\pi \circ \gamma)'(w)| = \lambda_\Omega(\pi(w)) \cdot |\pi'(w)|, \quad w \in D
\]

Next, for \( z \in \Omega \) set \( \delta_\Omega(z) = \inf\{ |\xi - z| : \xi \in \partial \Omega \} \) to be the Euclidean distance of point \( z \) to the boundary \( \partial \Omega \). We call \( \delta_\Omega(z)^{-1} |dz| \) the quasi-hyperbolic metric on \( \Omega \), which has many interesting applications in the geometric function theory, for example, [1] and [6].

We write \( A(\Omega), H(\Omega) \) and \( L^1_{\text{loc}}(\Omega) \) for the classes of analytic, real-valued harmonic and locally integrable functions on \( \Omega \), respectively.

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First, we define Bloch type spaces on $\Omega$.

The analytic Bloch space on $\Omega$ is given by

$$B(\Omega) = \{ f : f \in A(\Omega), \; \| f \|_{\partial \Omega} = \sup_{z \in \partial \Omega} \frac{|f'(z)|}{\lambda_\Omega(z)} < \infty \} \quad (1.2)$$

The analytic quasi-Bloch space on $\Omega$ is defined by

$$Q B(\Omega) = \{ f : f \in A(\Omega), \; \| f \|_{Q \partial \Omega} = \sup_{z \in \partial \Omega} \frac{|f'(z)|}{\delta_\Omega(z)} < \infty \} \quad (1.3)$$

Also, we use $B_{\alpha}(\Omega)$ and $QB_{\alpha}(\Omega)$ to stand for the spaces of harmonic Bloch and harmonic quasi-Bloch functions on $\Omega$, respectively as follows:

$$B_{\alpha}(\Omega) = \{ f : f \in H(\Omega), \; \| f \|_{\partial \Omega} = \sup_{z \in \partial \Omega} \frac{\nabla f(z)}{\alpha(z)} < \infty \}, \quad (1.4)$$

$$QB_{\alpha}(\Omega) = \{ f : f \in H(\Omega), \; \| f \|_{Q \partial \Omega} = \sup_{z \in \partial \Omega} \frac{\delta_\Omega(z)|\nabla f(z)|}{\delta_\Omega(z)} < \infty \}, \quad (1.5)$$

where $\nabla$ is the gradient operator.

Second, we define $BMO$ spaces on $\Omega$.

As far as we know, the area $BMO$ space on $\Omega$ is often given by

$$BMO(\Omega,m) = \{ f : f \in L^{1}_{\text{loc}}(\Omega), \; \| f \|_{m,\Omega} = \sup_{\Delta} \frac{1}{m(\Delta)} \int_{\Delta} |f(z) - f_{\Delta}| dm(z) < \infty \}, \quad (1.6)$$

where $dm(z)$ is the two-dimensional Lebesgue measure on $\Omega$,

$$f_{\Delta} = \frac{1}{m(\Delta)} \int_{\Delta} f(z) dm(z)$$

and

$$m(E) = \int_{E} dm(z), \quad E \subset \Omega.$$  

The supremum is taken over all Euclidean disks $\Delta \subset \Omega$.

Furthermore, we put $BMO_{A} (\Omega,m) = BMO(\Omega,m) \cap A(\Omega)$ and $BMO_{H} (\Omega,m) = BMO(\Omega,m) \cap H(\Omega)$.

Meanwhile, another area BMO space on $\Omega$ is defined by means of $BMO(D,m)$ and universal covering map $\pi$, namely,

$$BMO_{\pi} (\Omega,m) = \{ f : f \in L^{1}_{\text{loc}}(\Omega), \; f \circ \pi \in BMO(D,m), \; \| f \|_{\pi,\Omega,m} = \| f \circ \pi \|_{m,D} \}. \quad (1.7)$$

Moreover, let $BMO_{A} (\Omega,m) = BMO_{A} (\Omega,m) \cap A(\Omega)$ and $BMO_{H} (\Omega,m) = BMO(\Omega,m) \cap H(\Omega)$.

On the above spaces there have been some research, for instance, [1], [2], [3], [6], [7] and [9]. However those are incomplete. Following [7] we are about to give more deepgoing characterizations. For an application, we deal with relationships between these spaces and