THE BESOV AND TRIEBEL–LIZORKIN SPACES WITH HIGH ORDER ON THE LIPSCHITZ CURVES

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Abstract

The Besov spaces \( \dot{B}^{2,q}_p(\Gamma) \) and Triebel–Lizorkin spaces \( \dot{F}^{2,q}_p(\Gamma) \) with high order \( \varepsilon \in \mathbb{R} \) on a Lipschitz curve \( \Gamma \) are defined, when \( 1 < p \leq \infty \), \( 1 \leq q \leq \infty \). To compare to the classical case, a difference characterization of such spaces in the case \( |\varepsilon| < 1 \) is given also.

Introduction

Suppose that \( A : \mathbb{R} \to \mathbb{R} \) is a real-valued Lipschitz function, that is, there exists a constant \( M \) such that for all \( x,y \in \mathbb{R} \), \( |A(x) - A(y)| \leq M|x - y| \) and \( \Gamma \) is the graph of function \( x + iA(x) \) in the complex plane \( \mathbb{C} \). In [2] the Besov spaces \( \dot{B}^{2,q}_p \) and Triebel–Lizorkin spaces \( \dot{F}^{2,q}_p \) on \( \Gamma \) for \( -1 < \varepsilon < 1 \) and \( 1 \leq p \), \( q \leq \infty \) were introduced. In this note we introduce the Besov spaces \( \dot{B}^{2,q}_p \) and Triebel–Lizorkin spaces \( \dot{F}^{2,q}_p \) on \( \Gamma \) for \( |\varepsilon| \geq 1 \) and \( 1 \leq p \), \( q \leq \infty \). It has been known for some time that most of the function and distribution spaces on \( \mathbb{R}^n \), namely the Besov and Triebel–Lizorkin spaces, share a common underlying structure, and that by using Littlewood–Paley theory these spaces can be characterized through the action of appropriate families of convolution operators. Classically, the Fourier transform is the basic tool for studying Littlewood–Paley characterizations of these spaces. It is also well known that the Calderon reproducing formula ([1]) plays an important role in studying these spaces. For these and other facts about these spaces on \( \mathbb{R}^n \) the reader is referred to [4], [7] and [8]. The key idea to introduce the Besov and Triebel–Lizorkin spaces with high order on \( \Gamma \) is to establish the Calderon reproducing formula on \( \Gamma \) by using the

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higher order derivatives of the Cauchy kernel on $\Gamma$. It seems to need a new idea to generalize the results in this note to high dimensional Lipschitz surfaces.

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In this section we establish the Calderon reproducing formula on $\Gamma$. First we introduce the following definition.

**Definition (1.1).** Suppose that $0 < \beta \leq 1$, $\gamma > 0$ and $K \geq 0$ is an integer. A function $f: \mathbb{C} \to \mathbb{C}$ is said to be a $(\beta, \gamma, K)$ type smooth molecule centered at $w_0 \in \Gamma$ with the width $r > 0$ if there exists a constant $c$ such that

\begin{align*}
(1.1) & \quad |\frac{d^{v} f}{dw^{v}}(w)| \leq c \frac{r^{v}}{(r + |w - w_0|)^{v+\gamma+1}} \quad \text{for all } w \in \Gamma \text{ and } 0 \leq v \leq K, \\
(1.2) & \quad |\frac{d^{K} f}{dw^{K}}(w) - \frac{d^{K} f}{dw^{K}}(w')| \leq c \frac{r^{\gamma}}{(r + |w - w_0|)^{\gamma+\gamma+1}},
\end{align*}

for all $w$ and $w' \in \Gamma$ with $|w - w'| \leq \frac{1}{2}(r + |w - w_0|),$

\begin{align*}
(1.3) & \quad \int_{\Gamma} f(w)w^{\theta} dw = 0 \quad \text{for all } 0 \leq \theta \leq J \quad \text{where } J = [\gamma - 1], \text{ the least integer larger than or equal to } \gamma - 1.
\end{align*}

Denote $M^{(\beta, \gamma, K)}(w_0, r)$ the collection of all $(\beta, \gamma, K)$ type smooth molecules centered at $w_0 \in \Gamma$ with the width $r > 0$. If $f \in M^{(\beta, \gamma, K)}(w_0, r)$, the norm of $f$ in $M^{(\beta, \gamma, K)}(w_0, r)$ is defined to be

\begin{align*}
\|f\|_{M^{(\beta, \gamma, K)}(w_0, r)} = \inf\{c > 0, c \text{ satisfies (i) and (ii) in (1.2).}\}
\end{align*}

We denote $M^{(\beta, \gamma, K)}(iA(0), 1)$. It is easy to see that $M^{(\beta, \gamma, K)}$ is a Banach space under the norm $\|f\|_{M^{(\beta, \gamma, K)}} < \infty$. Just as the space of distributions $\mathcal{D}'$ is defined on $\mathbb{R}^n$, the dual space $(M^{(\beta, \gamma, K)})'$ consists of all linear functionals $\ell$ from $M^{(\beta, \gamma, K)}$ to $\mathbb{C}$ with the property that there exists a finite constant $c$ such that for all $f \in M^{(\beta, \gamma, K)}$, $|\ell(f)| \leq c \|f\|_{M^{(\beta, \gamma, K)}}$. We denote by $<h, f>$ the natural pairing of elements $h \in (M^{(\beta, \gamma, K)})'$ and $f \in M^{(\beta, \gamma, K)}$. It is easy to see that for $w_1 \in \Gamma$ and $d > 0$, $M^{(\beta, \gamma, K)}(w_1, d) = M^{(\beta, \gamma, K)}$ with equivalent norms. Thus, for all $h \in (M^{(\beta, \gamma, K)})'$ and all $f \in M^{(\beta, \gamma, K)}(w_1, d)$ with $w_1 \in \Gamma$ and $d > 0$, $<h, f>$ is well defined.

Now we establish the Calderon–type reproducing formula on $\Gamma$. For $f \in M^{(\beta, \gamma, K)}$, let

\begin{align*}
F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw, \quad z \in \mathbb{C} \setminus \Gamma, \text{ be the Cauchy integral of } f. \text{ Integrating by parts gives}
\end{align*}