A BEST APPROXIMATION PROPERTY OF DISCRETE QUADRATIC INTERPOLATORY SPLINES

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Abstract

We investigate the following problem in this paper: where there is an unique 1-periodic discrete quadratic spline \( s \in S(3,p,h) \) satisfying certain interpolatory condition for a 1-periodic discrete function defined on \([0,1]\). The answer is affirmative.

1. Introduction

Discrete splines were introduced by Mangasarian and Schumaker\[^6\], as a solution of certain discrete Best-interpolation problems. Schumaker\[^8\],\[^9\], Lyche\[^4\],\[^5\] and Astor and Duris\[^1\] have studied different constructive aspects of discrete splines and the problems relating dimension bases and interpolation and approximation powers of spaces of discrete splines have been investigated. Discrete splines are piecewise polynomial functions which satisfy smoothness requirements at the knots in terms of differences rather than derivatives. In particular discrete splines reduce to corresponding usual splines. Important interpolatory and convergence properties of discrete cubic splines have been studied by Dikshit and Powar\[^2\],\[^3\]. Rana\[^7\] has introduced discrete quadratic splines in terms of central differences and has obtained some interesting results concerning its interpolatory properties. In the present paper, we shall study the existence and convergence of a discrete quadratic spline which satisfies certain averaging interpolatory condition. Further we shall show that in a particular form, this interpolatory discrete quadratic spline satisfies a Best-approximation property with respect to certain functional. Similar Best-approximation property has been studied earlier by Sharma and Tzimbalario\[^10\] for the case of usual quadratic splines.

Given a real number \( h > 0 \) and any fixed real number \( a \), the discrete real line \( R_{ah} \) is given by

\[ R_{ah} = \{ \cdots, a-h, a, a+h, a+2h, \cdots \}, \]

and a discrete interval \([a,b]_h\) is defined by
Let $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\} \subset \mathbb{R}$ be a sequence of points in closed interval $[0,1]$. We denote the interval $[x_{i-1},x_i)$ by $I_i$, $i = 1,2,\ldots,n-1$ and interval $[x_{n-1},x_n]$ by $I_n$. Also the length of $i$th interval viz. $x_i - x_{i-1}$ is denoted by $p_i$ and $\bar{p}$ stands for $\max p_i$, $i = 1,2,\ldots,n$. A discrete quadratic spline on $[0,1]$ is a piecewise quadratic polynomial with knots $x_1,\ldots,x_{n-1}$ which satisfies following conditions:

(i) $s$ is continuous in $[0,1]$, so that $s_i(x_i) = s_{i+1}(x_i)$, where $s_i$ is the restriction of $s$ in $I_i$, $i = 1,2,\ldots,n-1$,

$$s_i(x_i) = s_{i+1}(x_i), \quad i = 1,2,\ldots,n-1.$$  

(ii) $D_h^{(1)} s_i(x_i) = D_h^{(1)} s_{i+1}(x_i + h)$, $i = 1,2,\ldots,n-1$.

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Here $D_h^{(r)}$ denotes the $r$th central difference operator which is defined by

$$D_h^{(0)} g(x) = g(x),$$

$$D_h^{(1)} g(x) = \frac{g(x+h) - g(x-h)}{2h},$$

$$D_h^{(2)} g(x) = \frac{g(x+h) + g(x-h) - 2g(x)}{h^2},$$

and so on.

We shall represent $D_h^{(r)} g(x)$ by $g^{(r)}(x)$, $r = 1,2,\ldots$. The space of discrete quadratic splines with knots $x_1,\ldots,x_{n-1}$ in $P$ is denoted by $S(3,P,h)$. We shall investigate the following:

**Problem 1.1.** Given a 1-periodic discrete function $f$ defined over $[0,1]$, and some real number $\alpha$, does there exist a unique 1-periodic discrete quadratic spline $s \in S(3,P,h)$ satisfying the interpolatory condition

$$s(x_{i-1}) + \alpha s(x_i) = f(x_{i-1}) + \alpha f(x_i), \quad i = 1,2,\ldots,n?$$  

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In Section 2, we settle the Problem 1.1. Convergence properties of the discrete quadratic spline of Theorem 2.1 will be established in Section 3. We denote the discrete quadratic spline of Theorem 2.1 with $\alpha = 1$ by $s_f$. In Section 4, we prove that $s_f$ admits a best approximation property with respect to a functional which we define in terms of forward differences. In fact, the $j$th forward differences of a function $g$ is given by

$$D_h^{(f)} g(x) = (1/h^j) \sum_{i=-j}^{j} (-1)^{j-i} \binom{j}{i} g(x + ih), \quad j = 0,1,2,\ldots.$$  

We shall denote the $j$th forward difference of a function $g$ by $g^{(j)}(x)$. Thus in particular we have

$$g^{(1)}(x) = \frac{g(x+h) - g(x)}{h};$$

$$g^{(2)}(x) = \frac{g(x+2h) - 2g(x+h) + g(x)}{h^2}.$$  

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