THE SIMULTANEOUS APPROXIMATION
IN THE LOCALLY CONVEX SPACES

Song Wenhua
(Dalian University of Technology, China)

Received Aug. 6, 1993 Revised June 6, 1994

Abstract

In this paper, we study the characterization of f-Chebyshev radius and f-Chebyshev centers and the existence of f-Chebyshev centers in locally convex spaces.

§ 1. Introduction

Let $X$ and $X'$ be a pair of linear spaces put in duality by a bilinear form $<,>$. We assume that this bilinear form $<,>$ is separating, i.e., for each $x \in X$ and $x \neq 0$, there exists $y$ in $X'$ such that $<x,y> \neq 0$ and, for each $y \in X'$ and $y \neq 0$, there exists $x \in X$ such that $<x,y> \neq 0$. A topology on $X$ is said to be compatible if it is a separated locally convex topology for which continuous linear functions on $X$ are precisely of the form

$$<*,y>: x \rightarrow <x,y>, \quad \text{for } y \in X'.$$

Let $f$ be a continuous convex function defined on $X$ and satisfying $f(0) = 0$. Given a nonempty subset $Y$ of $X$ and $x \in X$, let

$$f_Y(x) = \inf \{f(x - y); y \in Y\};$$

$$P_{f,Y}(x) = \{y \in Y; f_Y(x) = f(x - y)\};$$

The set-valued mapping $P_{f,Y}$ is called $f-$metric projection from $X$ onto $Y$. $Y$ is said to be $f$-proximinal (resp. $f$-Chebyshev) if $P_{f,Y}(x)$ is nonempty (resp. $P_{f,Y}(x)$ is a singleton) for each $x \in X$.

The concept of $f$-Chebyshev centers is introduced by D. V. Pai in [6].

Let $A$ be a bounded subset of $X$. We call the number

* Research supported by the National Science Foundation of P. R. China
to be the \( f \)-Chebyshev radius of \( A \) with respect to \( Y \). For \( y_0 \in Y \), if for every \( x \in A \), \( f(x - y_0) \leq r_{f,Y}(A) \), then we call \( y_0 \) to be an \( f \)-Chebyshev center of \( A \) with respect to \( Y \). We denote by \( Z_{f,Y}(A) \) the all \( f \)-Chebyshev centers of \( A \) with respect to \( Y \).

In this paper, we study the characterizations of \( f \)-Chebyshev radius and \( f \)-Chebyshev centers and the existence of \( f \)-Chebyshev centers.

We say that \( Y \) admits best simultaneous approximations with respect to \( A \) if \( Z_{f,Y}(A) \) is nonempty.

For \( r > 0 \), let
\[
S_r = \{ x \in X ; f(x) \leq r \}
\]
denote the sub—level subset of \( f \), and
\[
P_r(X) = \inf \{ \lambda > 0 ; x \in \lambda S_r \}
\]
denote the Minkowski gauge of \( S_r \). Then \( P_r \) is a non—negative continuous sublinear function.

Recall (cf. Govindarajulu and Pai[4]) that \( f \) is called to be infbounded if the sub—level sets \( S_r \) are bounded for each \( r > 0 \). A subset \( A \) is called to be \( f \)-bounded if for every \( x \in X \), one has
\[
\sup_{y \in A} |f(x-y)| < \infty.
\]

In this paper, we consider the following conditions:

\( (F1) \) There exists a continuous bijection \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that, for any \( x \in X \) and \( \lambda \geq 0 \), \( f(\lambda x) = \phi(\lambda)f(x) \) and \( f \) is continuous and convex.

\( (F2) \) \( f \) is a symmetric and sublinear function.

Obviously, if \( f \) satisfies the condition \( (F2) \), then \( f \) satisfies the condition \( (F1) \), and if there exists \( x \in X \setminus \{0\} \) such that \( f(x) > 0 \), then \( \phi \) is a convex function and \( \phi(t) \rightarrow \infty \) as \( t \rightarrow \infty \).

**Lemma P. G.** (D. V. Pai and P. Govindarajulu[6]) \( \text{Suppose } f \text{ satisfies the condition (F1) and } 0=f(0) \leq f(x). \text{ Then for any } \alpha, \beta > 0, \)
\[
S_\alpha = \frac{1}{\beta}S_{\phi(\beta)\alpha}, \quad P_\alpha = \beta P_{\phi(\beta)\alpha}.
\]
By this Lemma, we have
\[
P_{P_{f,Y}}(x) = P_{P_{f,Y}}(x),
\]
for any \( x \in X \) and \( \alpha, \beta > 0 \).

\( \S 2. \) \text{The case when } f \text{ satisfies the condition (F1)}

In this section, we assume that \( f \) satisfies the condition \( (F1) \).

Obviously, if there exists \( \lambda > 0 \) such \( f = \lambda g \), then