ORDER OF MEAN APPROXIMATION BY MIXED QUASI HERMITE–FEJER INTERPOLATION

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Received June 30, 1991 Revised Sep. 9, 1992

Abstract

In this paper we introduce a new kind of the mixed Hermite–Fejér interpolation with boundary conditions and obtain the mean approximation order. Our results include a new theorem of Varma and Prasad. Besides, we also get some other results about the mean approximation.

§ 1. Introduction

Let \( X_{n+2} = \{x_k = \cos\theta_k = \cos\frac{k\pi}{n+1} : 0 \leq k \leq n+1 \} \) be all zeros of the polynomial

\[
(1 - x^2) U_n(x) ,
\]

where \( U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta} \), \((x = \cos\theta, \theta \in [0,\pi])\) is the \( n \)-th Chebyshev polynomial of the second kind. For a non-negative integer \( r \), denote by \( C^r[-1,1] \) and \( \mathbb{R}(\mathbb{C}) \) the sets of the functions continuously differentiable to \( r \)-times and the real (complex) polynomials with degree at most \( r \), respectively. Let \( q, q' : 2q + 1 \geq q' \geq 0 \) be two integers. For \( f \in C^q[-1,1] \), consider the following interpolation problem:

\[
\begin{align*}
Q_{N_1}(f, x_k) &= f^{(j)}(x_k), 0 \leq j \leq q', 1 \leq k \leq n, \\
Q_{N_1}(f, x_k) &= 0, q' + 1 \leq j \leq 2q + 1, 1 \leq k \leq n, \\
Q_{N_1}(f, \pm 1) &= f^{(j)}(\pm 1), 0 \leq j \leq \min(q', q) = q_0, \\
Q_{N_1}(f, \pm 1) &= 0, q_0 + 1 \leq j \leq q, \\
Q_{N_1}(f) \in \prod_{N_1}(\mathbb{R}), N_1 = 2(q + 1)(n + 1) - 1.
\end{align*}
\]

It is well-known that there exists unique \( Q_{N_1}(f) \) satisfying the problem(1). We call \( Q_{N_1}(f) \) the mixed quasi Hermite–Fejér interpolation based on \( X_{n+2} \). Clearly, when
\[ q = q' = 0 \quad Q_{N_1}^{(f)}(f) = Q_{2n+1}^{(f)}(f) \] is just the usual quasi Hermite–Fejer interpolation\[3\], and moreover when \( q = 1 \) and \( q' = 0 \), \( Q_{N_1}^{(f)}(f) = \wedge_{n,2}^{(f,x)} \) is the so-called higher quasi Hermite–Fejer type interpolation, introduced by Sharma and Tzimbalario\[9\]. However, as far as we know, the other cases haven’t been studied.

There have been many works on the polynomial \( Q_{2n+1}^{(f)} \). In [3], Szasz proved that \( Q_{2n+1}^{(f)} \) converges uniformly on \([-1,1]\) to \( f \in C[-1,1] \). Later, the order of the convergence was given by Saxena and Mathur\[4\]:

\[
|Q_{2n+1}^{(f,x)}(f,x) - f(x)| = O\left(\frac{1}{n^{\alpha}} \sum_{k=1}^{n} \omega(f, \frac{1-x^2}{k}) + \frac{1}{k^2}\right).
\]  

(1.1)

Here and after, "\( \alpha \)" is independent of \( f \) and \( n \), \( \omega(f, \zeta) \) is the modulus of continuity of \( f \) on \([-1,1]\).

For \( \wedge_{n,2}^{(f,x)} \) in [9], it was proved that \( \wedge_{n,2}^{(f,x)} \) converges to \( f \in C[-1,1] \) uniformly on \([-1,1]\), however there was no estimate of approximation order. Recently, Varma and Prasad’s result\[11\] has stated that under the weighted \( L^p \) metrics, the approximation order of (1.1) can be improved by

\[
\|Q_{2n+1}^{(f)}(f) - f\|_{p,w} = O(\omega(f, \frac{1}{n})), \quad 0 < p < \infty.
\]  

(1.2)

Here for \( g \in C[-1,1], 0 < p < \infty \), denote \( \|g\|_{p,w} \) by

\[
\left(\int_{-1}^{1} |g(x)|^p w(x) dx\right)^{\frac{1}{p}}, \quad w(x) = (1-x^2)^{-\frac{1}{2}}.
\]

But, in [1], (1.2) was proved only for \( p = 2 \) and \( p = 4 \). Furthermore we can easily see that it is very difficult to show (1.2) for all \( p, 0 < p < \infty \) in view of [1]. So, how to prove (1.2) for all \( p, 0 < p < \infty \) is still an open problem. The main purpose of this paper is to answer this problem by using a new method. In fact we will claim the following two results:

**Theorem 1.** Let \( q, q' : 2q + 1 \geq q' \geq 0 \) be two integers and \( f \in C[-1,1] \), and let \( Q_{N_1}^{(f)}(f) \) be determined by (1). Then for all \( 0 < p < \infty \), and \( 0 \leq j \leq q' \) we have

\[
\|Q_{2n+1}^{(f)}(f,x) - \wedge_{n,2}^{(f,x)}(f,x)\|_{p,w} = O\left(n^{\frac{j+q'}{p}}(E_{N_1,2}^{(f,x)}(f,q') + (2q + 1 - q')\omega(f^{(q')}, \frac{1}{n}))\right)
\]  

(1.3)

Here for \( g \in C[-1,1], E_{r,g}(g) = \inf\{\|g - p\|_\infty : p \in \prod_{r}(R)\} \) and \( \|g\|_\infty = \max_{-1 < x < 1} |g(x)| \).

From this it follows that when \( q = q' = 0 \) (1.2) holds for all \( p, 0 < p < \infty \), by (1.3) of Theorem 1, and then the result of Varma–Prasad [1] as the above is included, and that when \( q = 1 \) and \( q' = 0 \), \( \wedge_{n,2}^{(f,x)}(f) \) has mean approximation order \( \omega(f, \frac{1}{n}) \).