ON THE LINEAR APPROXIMATION OPERATOR WHICH
HAS ALGEBRAIC PRECISION OF POINTED ORDER

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Received June 10, 1991 Revised Mar. 20, 1992

Abstract

In this paper, we construct an operator which has algebraic precision of pointed order and approximates
to \( f \in C[0,1] \) uniformly on \([0,1]\).

It is interesting to raise algebraic precision in linear approximation. We know that it reaches algebraic precision at most one order in positive linear approximation. Can we construct a linear approximation operator which has algebraic precision of order \( m \) for any fixed integer \( m \geq 2 \)? In the following, we give a definite answer for this question. The result is enlightened by the combinations of Bernstein operators (see Butzer [1], Chen Wenzhong [2], Ditzian [3]).

**Theorem.** For any integer \( m \geq 2 \), there exist constants \( c_i, d_i \) and function \( \lambda_i(n) \) of \( n \), where \( i = 1, 2, \ldots, p, p = (m + 1)(m + 2) / 2, \sum_{i=1}^{p} \lambda_i(n) \equiv 1 \) and for any \( 1 \leq i \leq p, \lambda_i(n) \) converges \((n \to \infty)\), such that the linear operator

\[
B_n(f,x) = \sum_{i=1}^{p} \lambda_i(n) \sum_{k=0}^{n} \binom{n}{k} f \left( k + c_i \frac{n^{(m-1)/m}}{n + d_i n^{(m-1)/m}} \right) x^k (1 - x)^{n-k}
\]

has algebraic precision of order \( m \), namely for any polynomial \( P_m(x) \) of degree \( m \), we have \( B_n(P_m,x) \equiv P_m(x) \) \((n \geq m, x \in [0,1])\), and \( B_n(f,x) \to f(x) \) \((n \to \infty)\) uniformly on \([0,1]\) for any \( f \in C[0,1] \).

**Proof.** Let \( B_n(f,x) \) be Bernstein polynomial

\[
B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) x^k (1 - x)^{n-k},
\]

* supported by the Science and Technology Fund of Shanxi Youth.
and

\[ B_{n,t}(f; t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n+t}\right) x^k (1-x)^{n-k}, \quad s \leq t, \]

then for any integers \( n \geq m \geq 2 \)

\[ B_n(x^m, x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n+t}\right) x^k (1-x)^{n-k} \]

\[ = \sum_{k=0}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} k^{m-1} \binom{n}{k} x^k (1-x)^{n-k} \]

\[ = \sum_{k=0}^{n} \frac{(n-1)!}{k!(n-k)!} k^{m-1} x^{k+1} (1-x)^{n-1-k} \]

\[ = \frac{x(n-1)^{n-1}}{m-1} B_{n-1}(x^m, x) + \binom{m}{1} \frac{(n-1)^{m-2}}{m-1} x B_{n-1}(x^m, x) + \cdots + \frac{1}{m-1} x \]

\[ = \frac{(n-1)(n-2)\ldots(n-m+1)}{m-1} x^{m} + O(1) \frac{1}{n} x^{m-1} + O(1) \frac{1}{n^2} x^{m-2} + \cdots + O(1) \frac{1}{n^{m-1}} x. \]

By inductive method we can prove that \( O(1)'s \) are bounded quantities, \( O(1) \neq o(1)(n \to \infty) \) and may be not equal to each other, then

\[ B_{n,t}(x^m, x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k+s}{n+t}\right) x^k (1-x)^{n-k} \]

\[ = \frac{1}{n+t} \sum_{k=0}^{n} \binom{n}{k} \left( k^{m} + \binom{m}{1} k^{m-1} s + \cdots s^{m} \right) x^k (1-x)^{n-k} \]

\[ = \frac{1}{n+t} \left\{ n^m B_n(x^m, x) + \binom{m}{1} n^{m-1} s B_n(x^m, x) + \cdots + \binom{m}{m-1} n^{m-1} B_n(x^m, x) + s^{m} \right\} \]

\[ = \frac{1}{n+t} \left\{ n \left[ \frac{(n-1)(n-2)\ldots(n-m+1)}{n^{m-1}} x^{m} + O(1) \frac{1}{n} x^{m-1} + \cdots + O(1) \frac{1}{n^{m-1}} x \right] \right. \]

\[ + \binom{m}{1} n^{m-1} s \left[ O(1) x^{m-1} + O(1) \frac{1}{n} x^{m-2} + \cdots + O(1) \frac{1}{n^{m-1}} x \right] + \cdots + \binom{m}{m-1} n^{m-1} s^{m-1} \]

\[ = \frac{1}{n+t} \left\{ n(n-1)\ldots(n-m+1)x^{m} + n^{m-1}(O(1) + O(1)s)x^{m-1} \right. \]

\[ + n^{m-2}(O(1) + O(1)s + O(1)s^2)x^{m-2} + \cdots + n(O(1) + O(1)s + \cdots + O(1)s^{m-1})x + s^{m} \} \right\}. \]

Let

\[ B_n^{*}(f,x) = \sum_{i=1}^{s} \lambda_i(n) B_{n,i} f(x), \]

\[ B_{n,t}(f,x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k+s}{n+t}\right) x^k (1-x)^{n-k}, \quad s \leq t, \quad i = 1, 2, \ldots, p, \]