A NOTE ON MODIFIED BASKAKOV TYPE OPERATORS *

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Abstract

In the present paper, we define a new kind of positive linear operators and study the rate of convergence in simultaneous approximation. This operator being capable of providing better approximation than modified Baskakov operators.

1. Introduction

Durrmeyer [5] introduced the integral modification of Bernstein polynomials, Derriennic [3] studied these operators and obtained some direct theorems. Recently Chen [2] introduced a general summation–integral type operator which generalizes e.g. Derriennic’s operator or its Szász–Mirakyan variant. A lot of work has been done on Durrmeyer–type operators (see e.g. [1], [4], and [6]–[10] etc.).

We define another modification of Baskakov operators by taking the weight function of Beta operators on $L_1[0,\infty)$ as

$$L_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^\infty b_{n,k}(t)f(t)dt, \quad x \in [0,\infty),$$

(1.1)

where

$$p_{n,k}(x) = \binom{n+k-1}{k} x^k / (1+x)^{n+k} \quad \text{and} \quad b_{n,k}(t) = t^k / B(k+1,n)(1+t)^{n+k+1}$$

(1.2)

$B(k+1,n)$ being the Beta function given by $k! (n-1)! / (n+k)!$.

By $H_0[0,\infty)$, we denote the class of all measurable functions defined on $[0,\infty)$ satisfying:

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\[ \int_0^\infty \frac{|f(t)|}{(1+t)^{n+1}} \, dt < \infty \] for some positive integer \( n \).

This class is bigger than the class of all Lebesgue integrable functions on \([0, \infty)\).

In this paper, we study the simultaneously approximation by the operators (1.1). To prove the main results, we shall need the following lemmas:

**Lemma 1.** For \( m \in \mathbb{N} \setminus \{0\} \), if

\[
U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left( \frac{k}{n} - x \right)^m,
\]

then

\[
U_{n,0}(x) = 1, \quad U_{n,1}(x) = 0, \\
nU_{n,m+1}(x) = x(1+x)[U'_{n,m}(x) + mU_{n,m-1}(x)].
\]

Consequently,

\[
U_{n,m}(x) = O\left(n^{-\left(\frac{m+1}{2}\right)}\right),
\]

where \([\beta]\) denotes the integral part of \( \beta \).

**Lemma 2.** There exist the polynomials \( q_{i,j,r}(x) \) independent of \( n \) and \( k \) such that

\[
x^r(1+x)^s \frac{d^r}{dx^r} (x^k(1+x)^{-s-k}) = \sum_{i+j \geq r} n^i (k-nx)^j q_{i,j,r}(x) x^k(1+x)^{-s-k}.
\]

**Lemma 3.** For \( r \in \mathbb{N} \setminus \{0\} \), we define

\[
V_{r,n,m}(x) = \sum_{k=0}^{\infty} p_{n+n,r,k}(x) \int_0^x b_{n-r,k+r}(t)(t-x)^m \, dt.
\]

Then

\[
V_{r,n,0}(x) = 1, \quad V_{r,n,1}(x) = \frac{1+ r + x(1+2r)}{n-r-1}, \quad n > r + 1
\]

\[
V_{r,n,2}(x) = \frac{2(2r^2 + 4r + n + 1)x^2 + 2(2r^2 + 5r + n + 2)x + (r^2 + 3r + 2)}{(n-r-1)(n-r-2)},
\]

and there holds the recurrence relation

\[
(n-m-r-1) V_{r,n,m+1}(x) = x(1+x)[V'_{r,n,m}(x) + 2m V_{r,n,m-1}(x)] + [(m+r+1)
\]

\[
(1+2x) - x] V_{r,n,m}(x), \quad n > m + r + 1.
\]

Consequently, for each \( x \in [0, \infty) \)

\[
V_{r,n,m}(x) = O\left(n^{-\left(\frac{m+1}{2}\right)}\right).
\]