SOME REMARKS ON OVERCONVERGENCE OF HERMITE INTERPOLATING POLYNOMIALS *

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Abstract

In this paper, a quantitative estimate for Hermite interpolant to function \( \psi(z) = (z^n - \beta^n)' \) on the zeros of \((z^n - \alpha^n)'\) is obtained. Using this estimate, a rather wide extension of the theorem of Walsh is proved and five special cases of it are given.

1. Introduction

Let \( A_\rho \) denote the class of functions analytic in the disc \( D_\rho = \{ z : |z| < \rho \} \) but not on \( |z| = \rho, \) where \( \rho > 0. \) For any \( f \in A_\rho, \alpha \in D_\rho, \) and \( n \in \mathbb{N} \) we denote by \( L_{n-1}(z; \alpha, f) \) the Lagrange interpolant to \( f \) on the zeros of \( z^n - \alpha^n. \) In 1986, Lou\(^{[1]} \) proved

Theorem A. If \( m = rn + q, r \in \mathbb{N}, 0 \leq s \leq q / n < 1 \) and \( q / n = s + O(\frac{1}{n}). \) Then for any \( f(z) \in A_\rho \) and \( \alpha, \beta \in D_\rho, \) we have

\[
\lim_{s \to 0} \Delta_{n,m}^{a,b}(z; f) = \lim_{s \to 0} \left\{ L_{n-1}(z; \alpha, f) - L_{n-1}(z; \alpha, L_{n-1}(z; \beta, f)) \right\}
= 0 \quad |z| < \sigma,
\]

where

\[
s = \rho / \max\left\{ \frac{|\alpha|}{\rho}, \frac{|\beta|}{\rho} \right\}^{r+s}.
\]

More precisely, for any \( \mu \) with \( \rho \leq \mu < \infty, \) we have

\[
\lim_{s \to 0} \left\{ \max_{\rho} |\Delta_{n,m}^{a,b}(z; f)| \right\}^{1/n} \leq \mu / \sigma.
\]

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Moreover, if $\alpha, \beta, m$ satisfy neither $\alpha = \beta = 0$ nor $\alpha' = \beta'$ when $m = rn$, then (1.1) is best possible in the sense that for any $z_0$ with $|z_0| = \sigma$, there is a function $f_0 \in A_{\rho}$ such that (1.1) does not hold at $z_0$.

The above result not only gives a result of Cavaretta, Sharma and Varga\textsuperscript{[2]} which itself is a generalization of a Walsh's Theorem, when $\alpha = 1, \beta = 0$ and $m = rn$, but also gives a theorem of Rivlin\textsuperscript{[3]} when $\alpha = 0, \beta = 1$ and $s = 0$.

Recently, there are some extensions of Theorem A to Hermite interpolation. Let $h_{r-1}(z; \alpha, f)$ denote the Hermite interpolant to $f$ on the zeros of $(z^n - x^n)'$, where $r$ is a positive integer and $x \in D_{\rho}$. In 1990, Akhlaghi, Jakimovshi and Sharma\textsuperscript{[4]} proved the following results:

**Theorem B.** Let $\alpha, \beta \in D_{\rho}$ and let $r > p$ be positive integers. Then for any $f \in A_{\rho}$, we have

$$
\lim_{n \to \infty} H_{\rho/n, r}(z; f) = \lim_{n \to \infty} \left\{ h_{pn-1}(z; \alpha, f) - h_{rn-1}(z; \alpha, h_{rn-1}(z; \beta, f)) \right\} = 0, \quad |z| < \sigma
$$

where

$$
\sigma = \rho / \max\left[ |x|^{(r+1-p)/p}, |r|^{(r+1-p)/p} \right],
$$

the convergence being uniform and geometric in $|z| < R < \sigma$. Moreover the result is best possible in the sense the (1.4) does not hold for every point on $|z| = \sigma$ and for every $f \in A_{\rho}$.

**Theorem C.** If $\alpha \neq \beta \in D_{\rho}$ and $r = sp + q, 0 \leq q < p - 1$, then for any $f \in A_{\rho}$, we have

$$
\lim_{n \to \infty} \Lambda_{\rho/n, r}(z; f) = \lim_{n \to \infty} \left\{ L_{pn-1}(z; \alpha, f) - L_{rn-1}(z; \alpha, h_{rn-1}(z; \beta, f)) \right\} = 0, \quad |z| < \sigma,
$$

where

$$
\sigma = \begin{cases} 
\rho / \max\left[ |x|^{r-s}, |x|^{l-s+1+q}/p \right], & \text{if } |x| < |\beta|, \\
\rho / \max\left[ |x|^{r-s}, |x|^{l-s+1+q}/p \right], & \text{if } |x| > |\beta|,
\end{cases}
$$

the convergence being uniform and geometric in $|z| < R < \sigma$. Moreover the result is best possible in the sense that for any $z_0$ with $|z_0| = \sigma$, there is a function $\tilde{f}$ such that (1.6) does not hold for $\tilde{f}$ at $z_0$.

**Theorem D.** If $m = rn + q, r \geq p, q = ns + O(1), 0 \leq s < 1$, and $\alpha \neq \beta \in D_{\rho}$, then for any $f \in A_{\rho}$, we have