TWO WEIGHT INEQUALITIES FOR FRACTIONAL ONE-SIDED MAXIMAL OPERATORS ON ORLICZ AND LORENTZ SPACES

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Abstract

In the paper, we characterize two weight weak and strong type inequalities for the fractional one-sided maximal operators on Lorentz and Orlicz space.

1 Introduction

In [1], Sawyer introduced the one-sided Hardy-Littlewood maximal functions and characterized the inequalities from $L^p(u)$ to $L^q(v)$ ($1 \leq p \leq q < \infty$), in [2], the fractional one-sided maximal operators are introduced and the weighted Lebesgue norm inequalities are considered. In this paper, we characterize the weighted weak and strong type inequalities for the fractional one-sided maximal operators on Lorentz and Orlicz spaces.

Let $0 < a < 1$, following [1][2], the fractional one-sided maximal operator is defined as

$$M_a^+ f(x) = \sup_{s > 0} \frac{1}{h_s^a} \int_{h_s^a} f(y) \, dy.$$  \hspace{1cm} (1)

Now we begin with a brief summary of the facts about Lorentz and Orlicz spaces, see [3][4][5] for the details.

Suppose $u$ is a weight function on $\mathbb{R}^1$, for a measurable function $f$ on $\mathbb{R}^1$, let

$$u_f(\lambda) = u(\{x \in \mathbb{R}^1 : |f(x)| > \lambda\}), \quad \lambda > 0,$$

and

$$f^*(t) = \inf\{\lambda > 0 : u_f(\lambda) \leq t\},$$

the Lorentz space $L^N(u)$ consists of all measurable functions $f$ satisfying $\|f\|_{L^N(u)} < \infty$, where

$$\|f\|_{L^N(u)} = \left\{ \begin{array}{ll}
\left( \frac{Q}{p} \int_0^\infty (t^{1/p} f^*(t)^q \frac{dt}{t})^{1/q} \right)^{1/p}, & 0 < p, q < \infty, \\
\sup_{t > 0} t^{1/p} f^*(t), & 0 < p \leq \infty, \quad q = \infty.
\end{array} \right.$$  

The following lemma see [4].
Lemma 1  (1) If $\chi_E$ denotes the characteristic function of the set $E$, then
$$\| \chi_E \|_{L^r(u)} = u(E)^{1/r};$$
(2) If $1<p<\infty$, $1\leq q<\infty$ or $p=q=1$ or $p=q=\infty$, then
$$C_1 \| f \|_{L^r(u)} \leq \sup \{ \| \int gudx \|_{L^r(u)} : \| g \|_{L^r(u)} \leq 1 \} \leq C_2 \| f \|_{L^r(u)};$$
(3) If $1\leq p\leq q<\infty$ and $\{ E_j \}$ are a sequence of disjoint measurable subsets of $\mathbb{R}^1$, then
$$\sum_j \| f \chi_{E_j} \|_{L^r(u)} \leq C \| f \|_{L^r(u)};$$
(4)
$$\| f \|_{L^r(u)} = \begin{cases} \left( \int_0^\infty q^{1/p} u_j(\lambda)^{1/q} d\lambda \right)^{1/p}, & p, q < \infty, \\ \sup_{\lambda>0} \lambda u_j(\lambda)^{1/p}, & p < \infty, q = \infty. \end{cases}$$

Suppose $\Phi$ is the Young function, i.e. $\Phi(t) = \int_0^t \varphi(s)ds$, where $\varphi$ is nondecreasing right continuous function defined on $\mathbb{R}^+$ with $\varphi(0^+)=0, \varphi(\infty)=\infty$. Let $\Psi$ be the complementary function of $\Phi$, i.e. $\Psi(t) = \int_0^t \varphi^{-1}(s)ds$, where $\varphi^{-1}$ is the right continuous inverse function of $\varphi$. We have
$$\Phi(\Psi(t)/t) \leq \Psi(t), \quad t \geq 0. \quad (2)$$

The Orlicz space $L_\Phi(u)$ consists of all measurable functions $f$ such that
$$\| f \|_{\Phi(u)} = \inf \{ \lambda > 0 : \int \Phi(\| f \|/\lambda)udx \leq 1 \} < \infty \quad (3)$$

A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition (we write this as $\Phi \in \Delta_2$) if
$$\Phi(2t) \leq c\Phi(t), \quad \text{for all } t \geq 0; \quad (4)$$
If $\Phi \in \Delta_2$, then there exists $a, \beta$ such that $1 \leq \beta \leq a < \infty$ and
$$\lambda^\alpha \Phi(t) \leq \Phi(\lambda t) \leq \lambda^\beta \Phi(t), \quad \text{for } \lambda \geq 1, t \geq 0$$
$$\lambda^{\alpha'} \Phi(t) \leq \Phi(\lambda t) \leq \lambda^{\beta'} \Phi(t), \quad \text{for } \lambda < 1, t \geq 0. \quad (5)$$

Definition 1  Suppose $\Phi_1, \Phi_2$ are Young functions, we denote $\Phi_1 \ll \Phi_2$ if for every sequence $(a_n)$, the following inequality holds
$$\sum_n \Phi_1(\Phi_2^{-1}(|a_n|)) \leq C\Phi_2(\Phi_1^{-1}(\sum_n |a_n|)).$$

2 L$^r$-inequalities for $\text{M}_a^+$

Theorem 1  Let $1\leq s \leq p \leq q < \infty$ and $u, v$ be weight functions on $\mathbb{R}^1$, then the following statements are equivalent:

(A) $\| M^*_sf \|_{L^r(u)} \leq C \| f \|_{L^r(u)}$, for all $f \in L^r(u)$;
(B) $v(I_1)^{1/q} \| \chi_{I_1}/u \|_{L^r(u)} \leq C |I|^s$, for any interval $I=I_1, I_2=[a, b)$ and $x \in (a, b), I_1=[a, x), I_2=[x, b)$.

Theorem 2  Let $1 \leq p \leq q < \infty$ and $u, v$ be weight functions on $\mathbb{R}^1$, then the following