SIMULTANEOUS APPROXIMATION TO
A DIFFERENTIABLE FUNCTION AND ITS DERIVATIVES
BY LAGRANGE INTERPOLATING POLYNOMIALS*

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Abstract

This paper establishes the following pointwise result for simultaneous Lagrange interpolating approximation: Let \( f \in C^{q}_{[-1,1]} \) and \( r = \left\lfloor \frac{q + 2}{2} \right\rfloor \), then
\[
\left| f^{(k)}(x) - P_n^{(k)}(f,x) \right| = O(1) \Delta_n^{q-k}(x) \| f \|_q + \| f \|_q \Delta_n^{q-k}(x), \quad 0 \leq k \leq q,
\]
where \( P_n(f,x) \) is the Lagrange interpolating polynomial of degree \( n + 2r - 1 \) of \( f(x) \) on the nodes \( X_n \cup Y_n \) (see the definition of the next). \( \Delta_n(x) = \frac{1-x^2}{n} + \frac{1}{n^2} \).

1. Introduction

In recent years, many scholars published a lot of papers on simultaneous approximation to differentiable functions and their derivatives by interpolating polynomials (see [2], [5], [7], [9] etc.). Among these works, [2] proved the following interesting result. Let \( c > 0 \) be a given real number,
\[
X_n := \{-1 < x_n < x_{n-1} < \cdots < x_1 < 1\},
\]
\[
Y_n := \{-1 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{r-1} \leq -1 + c / n^2, 1 - c / n^2 \leq t_r \leq \cdots \leq t_{2r-2} \leq s_{2r-1} = 1\}.
\]

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Write

\[ l_k = \frac{\omega_n(x)}{\omega_n'(x_k)(x - x_k)}, \]

where

\[ \omega_n(x) = \prod_{i=1}^{n}(x - x_i), \]

\[ \|L_n\| = \max_{|x| < 1} \sum_{k=1}^{n} |l_k(x)|, \]

\[ \|L_n^*\| = \max_{|x| < 1} \sum_{k=1}^{n} \frac{1 - x_k^2}{1 - x_k^2} |l_k(x)|. \]

For \( f \in C_{[-1,1]}^q \) (that is, \( f(x) \) has \( q \) continuous derivatives on \([-1,1]\)), let \( P_n(f,x) \) denote the Lagrange interpolating polynomial of degree \( n + 2r - 1 \) of \( f \) on the nodes \( X_n \cup Y_n \). When \( X_n \cup Y_n \) has coalescing nodes, the derivatives of \( P_n(f,x) \) interpolate the derivatives of \( f(x) \) on these nodes with the same multiplicity, that is, for example, if \( t_i = t_{i+1} = \cdots = t_{i+q} \), then \( f^{(i)}(t_j) = P_n^{(i)}(t_j), i = 0,1,\ldots,q. \)

With all the above notations, [2] proved the following theorem BK.

**Theorem BK.** Let \( f \in C_{[-1,1]}^q, r = \left[ \frac{q+1}{2} \right] \). Then for \( k = 0,1,\ldots,q \), the following estimates hold:

a) \( \|f^{(k)}(x) - P_n^{(k)}(f,x)\| = O(1)n^{-q-k}w(f^{(q)}_{\text{max}})\|L_n\| \), when \( q \) is an even number;

b) \( \|f^{(k)}(x) - P_n^{(k)}(f,x)\| = O(1)n^{-q-k-1}w(f^{(q)}_{\text{max}})\|L_n\| \), when \( q \) is an odd number;

c) \( \|f^{(k)}(x) - P_n^{(k)}(f,x)\| = O(1)n^{-q-k}w(f^{(q)}_{\text{max}})\|L_n^*\| \), when \( q \) is an odd number,

where \( w(f,t) \) is the modulus of continuity of \( f(x) \).

It is well-known that \( \|L_n\| \geq \frac{\log n}{8\sqrt{\pi}} \). However when \( X_n \) is taken to be the zeros of the Chebyshev polynomial \( T_n(x) = \cos(n\arccos x) \), one has that \( \|L_n\| = O(1)\log n \). Also, one has \( \|L_n^*\| = O(1)\log n \) in this case. Therefore, Theorem BK contains those results of [7] and [8] as well as [1] as special cases.

We already knew that, in the approximation of continuous functions by polynomials, there are often better estimates on the endpoints of the interval considered. For example, [10] showed that, for \( f \in C_{[-1,1]}^q \), there are polynomials \( p_n(x) \) of degree \( n \) such that...