Cohomology of the moduli of parabolic vector bundles

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Abstract. The purpose of this paper is to compute the Betti numbers of the moduli space of parabolic vector bundles on a curve (see Seshadri [7], [8] and Mehta & Seshadri [4]), in the case where every semi-stable parabolic bundle is necessarily stable. We do this by generalizing the method of Atiyah and Bott [1] in the case of moduli of ordinary vector bundles. Recall that (see Seshadri [7]) the underlying topological space of the moduli of parabolic vector bundles is the space of equivalence classes of certain unitary representations of a discrete subgroup \Gamma which is a lattice in PSL(2, R). (The lattice \Gamma need not necessarily be co-compact).

While the structure of the proof is essentially the same as that of Atiyah and Bott, there are some difficulties of a technical nature in the parabolic case. For instance the Harder-Narasimhan stratification has to be further refined in order to get the connected strata. These connected strata turn out to have different codimensions even when they are part of the same Harder-Narasimhan strata.

If in addition to 'stable = semistable' the rank and degree are coprime, then the moduli space turns out to be torsion-free in its cohomology.

The arrangement of the paper is as follows. In § 1 we prove the necessary basic results about algebraic families of parabolic bundles. These are generalizations of the corresponding results proved by Shatz [9]. Following this, in § 2 we generalize the analytical part of the argument of Atiyah and Bott (§ 14 of [1]). Finally in § 3 we show how to obtain an inductive formula for the Betti numbers of the moduli space. We illustrate our method by computing explicitly the Betti numbers in the special case of rank = 2, and one parabolic point.

Keywords. Cohomology; parabolic vector bundles; moduli space; Betti numbers; algebraic family; Sobolev spaces.

1. Algebraic families of parabolic bundles

For the basic definitions and properties of vector bundles with parabolic structures ('parabolic bundles' for short) see Seshadri [7] and [8], and Mehta and Seshadri [4]. For simplicity we assume throughout that the parabolic structures of the parabolic bundles considered are over only a single point \( P \) of the curve \( X \). All our arguments generalize trivially to the case where there are more parabolic points on \( X \).

As in [8], each parabolic bundle \( E \) on \( X \) has a unique (parabolic) Harder-Narasimhan filtration \( 0 \subset E_1 \subset \ldots \subset E_r = E \), where \( E_i \) are parabolic subbundles such that (i) each quotient parabolic bundle \( E_i/E_{i-1} \) is semi-stable (ii) the inequality \( \text{par} \mu(E_i/E_{i-1}) > \text{par} \mu(E_{i+1}/E_i) \) holds for each \( i \), where \( \text{par} \mu = (\text{par deg}/\text{rank}) \).
Hence to each parabolic bundle $E$, we may associate a convex polygon $\text{HNP}(E)$ in $\mathbb{R}^2$ as defined by Shatz [9]. The vertices of $\text{HNP}(E)$ (= the Harder–Narasimhan polygon of $E$) are the points $(0,0), (\text{rank } E_1, \text{par deg } E_1), \ldots, (\text{rank } E_r, \text{par deg } E_r)$. The polygons corresponding to parabolic bundles of a given rank and parabolic degree have a natural partial ordering as defined by Shatz [9], namely if $\lambda_1$ and $\lambda_2$ are two such polygons then $\lambda_1 \geq \lambda_2$ if all the vertices of $\lambda_2$ lie on or below the polygon $\lambda_1$. If $\lambda_1 \geq \lambda_2$, we say that $\lambda_1$ dominates $\lambda_2$. The polygon $\lambda = \text{HNP}(E)$ is also called the Harder–Narasimhan type of the parabolic bundle $E$.

**PROPOSITION 1.1.** Let $E$ be a parabolic bundle. Then any parabolic subbundle of $E$ lies on or below the Harder–Narasimhan polygon of $E$. In particular, the polygon corresponding to any filtration of $E$ by parabolic subbundles is dominated by the Harder–Narasimhan polygon $\text{HNP}(E)$ of $E$.

**Proof.** If $F_1$ and $F_2$ are parabolic subbundles of $E$, and if the vector subbundles $F_1 \vee F_2$ and $F_1 \cap F_2$ of $E$ (as defined in Langton [3]) are given the induced parabolic structure, then it is easy to see that $\text{par deg } (F_1 \vee F_2) + \text{par deg } (F_1 \cap F_2) \geq \text{par deg } (F_1) + \text{par deg } (F_2)$. Now the proof of proposition 2 follows from the proof of the theorem 2 of Shatz [9].

We want to study how the Harder–Narasimhan type changes within an algebraic family of parabolic bundles. For this, the following observation is basic.

**Remark 1.2.** Let $E$ be a vector bundle on a scheme $S$ and let $F_1$ and $F_2$ be subbundles. Let $\phi: F_1 \to E/F_2$ be the natural map. Then the function $s \mapsto \text{rank}_{k(s)}(\phi(s))$ is lower-semicontinuous on $S$. (Equivalently, the function $s \mapsto \text{dim}_{k(s)}(F_1(s) \cap F_2(s))$ is upper semicontinuous).

Using this, it is easy to prove the following:

**PROPOSITION 1.3.** Let $E_T$ be a family of parabolic bundles parametrized by the scheme $T$ which is the spectrum of a discrete valuation ring. Let $\xi$ and $\xi_0$ be the generic and special points of $T$. Let $G$ be a coherent torsion-free quotient sheaf of $E_T$ on $X \times T$ which is flat over $T$. Let $G_\xi$ and $G_{\xi_0}$ be the restrictions of $G$ to $X \times \xi$ and $X \times \xi_0$. Let $G'_{\xi}$ be the vector bundle on $X \times \xi_0$ generically generated by $G_{\xi}$, $G'_{\xi_0}$ be the vector bundle on $X \times \xi_0$ generically generated by $G'_{\xi}$. Let $G_\xi$ and $G_{\xi_0}$ be given the induced parabolic structures, as quotients of $E_\xi$ and $E_{\xi_0}$ respectively. Then $\text{par deg}(G_\xi) \geq \text{par deg}(G_{\xi_0})$.

**PROPOSITION 1.4.** Let $E_T$ be a family of parabolic bundles on $X$ parametrized by $T$, where $T$ is the spectrum of a discrete valuation ring. Let $\xi$, $\xi_0$ be the generic and special points of $T$. Then $\text{HNP}(E_{\xi_0}) \geq \text{HNP}(E_\xi)$.

**Proof.** Let $0 \subset E_{\xi,1} \subset \ldots \subset E_{\xi,r} = E_\xi$ be the Harder–Narasimhan filtration over the generic point $\xi \in T$. Then using completeness of appropriate quot schemes it follows (Shatz [9], proposition 9) that there exists a filtration $0 \subset E_1 \subset \ldots \subset E$ of the bundle $E \to T \times X$ by subbundles (i.e. torsion free, coherent subsheaves