Inverse theory of Schrödinger matrices

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Abstract. In this note we discuss the inverse spectral theory for Schrödinger matrices, in particular a conjecture of Gesztesy-Simon [1] on the number of distinct iso-spectral Schrödinger matrices. We consider 3 x 3 matrices and obtain counter examples to their conjecture.

Keywords. Schrödinger matrices; iso-spectral sets; inverse spectral theory.

1. Introduction

Given a point $B = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$, consider the Schrödinger matrix $H(B)$ as

$$
H(B) = \begin{pmatrix}
  b_1 & 1 & \cdots & 0 \\
  1 & b_2 & 1 & \cdots & 0 \\
  0 & 1 & \ddots & 1 & 0 \\
  0 & \cdots & 1 & \ddots & 1 \\
  0 & \cdots & \cdots & 1 & b_n
\end{pmatrix}.
$$

This matrix has all real and distinct eigenvalues say $t_1, \ldots, t_n$. Define the map $F : \mathbb{R}^n \to \mathbb{R}^n$ as $F(B) = (t_1, \ldots, t_n)$ where $t_1 < \cdots < t_n$ are the eigenvalues of $H(B)$. Let $W_n$ denote the range of $F$, then the following conjecture is posed in [1].

Conjecture 1.1 (Gesztesy–Simon). Let $n$ be a positive integer and let $F$ and $W_n$ be as above. Then

1. $W_n$ is closed set in $\mathbb{R}^n$ whose interior is dense in $W_n$. For $(t_1, \ldots, t_n)$ in interior of $W_n$, $F^{-1}(t_1, \ldots, t_n)$ contains $n!$ points and for $(t_1, \ldots, t_n)$ in $\partial W_n$, $F^{-1}(t_1, \ldots, t_n)$ contains fewer than $n!$ points.
2. $F^{-1}(W_n^{\text{int}})$ is a disjoint union of $n!$ sets and on each of them $F$ is a diffeomorphism to $(W_n^{\text{int}})$.

We address (1) of the above conjectures. That $W_n$ is closed is already proved in [1] (theorem 7.4).

The case $n = 2$ is solved in the paper [1]. Now we will inspect the map $F$ for the case $n = 3$. Given a point $B \in \mathbb{R}^3$, consider the characteristic polynomial of $H(B)$, given by $\det(H(B) - tI) = 0$, viz.,

$$
t^3 - (b_1 + b_2 + b_3)t^2 + (b_1b_2 + b_2b_3 + b_1b_3 - 2)t + b_1 + b_3 - b_1b_2b_3 = 0.
$$

Observe that the characteristic polynomial is not symmetric in $b_1, b_2, b_3$. 

Lemma 1.2. For \( T = (t_1, t_2, t_3) \in \mathbb{R}^3 \), define \( P_1 = t_1 + t_2 + t_3; \ P_2 = t_1 t_2 + t_2 t_3 + t_1 t_3; \ P_3 = t_1 t_2 t_3 \). Then \( (t_1, t_2, t_3) = F(b_1, 0, b_3) = F(b_3, 0, b_1) \) iff \( (t_1, t_2, t_3) \) satisfy
\[
(P_1^2 - 4(2 + P_2)) \geq 0 \quad \text{and} \quad P_1 + P_3 = 0. \tag{1}
\]

Proof. Consider the characteristic polynomial of \( H(B) \) for \( B = (b_1, 0, b_3) \). Express the relations between the coefficients of that polynomial and its roots \( t_1, t_2, t_3 \). Now the lemma is a straightforward consequence. For the converse part set
\[
b_1 = \frac{P_1 \pm \sqrt{P_1^2 - 4(2 + P_2)}}{2}; \quad b_3 = \frac{P_1 \pm \sqrt{P_1^2 - 4(2 + P_2)}}{2}
\]
and again verify those relations. \( \square \)

Theorem 1.3. For \( (t_1, t_2, t_3) \in \mathbb{R}^3 \) and \( P_1, P_2, P_3 \) as defined in lemma (1.2) we have, \( (t_1, t_2, t_3) \in \text{Range } (F) \) iff the following system of equations has a real solution
\[
y^3 - P_1 y^2 + (3 + P_2) y - (P_1 + P_3) = 0 \quad 3y^2 + 2P_1 y + P_1^2 - 4(2 + P_2) \geq 0. \tag{2}
\]

Proof. Observe that \( (t_1, t_2, t_3) = F(b_1, b_2, b_3) \) iff
\[
(t_1 - b_2, t_2 - b_2, t_3 - b_2) = F(b_1 - b_2, 0, b_3 - b_2). \tag{3}
\]
Let
\[
\bar{P}_1 = t_1 - b_2 + t_2 - b_2 + t_3 - b_2 = P_1 - 3b_2,
\bar{P}_2 = (t_1 - b_2)(t_2 - b_2) + (t_3 - b_2)(t_2 - b_2) + (t_1 - b_2)(t_3 - b_2)
= P_2 - 2P_1 b_2 + 3b_2^2,
\bar{P}_3 = (t_1 - b_2)(t_2 - b_2)(t_3 - b_2) = P_3 - P_2 b_2 + P_1 b_2^2 - b_2^3.
\]
By lemma (1.2), eq. (3) holds iff \( (t_1, t_2, t_3) \) satisfy
\[
(\bar{P}_1^2 - 4(2 + \bar{P}_2)) \geq 0 \quad \text{and} \quad \bar{P}_1 + \bar{P}_3 = 0. \tag{4}
\]
The conditions (4) for \( \bar{P}_1, \bar{P}_2, \bar{P}_3 \) translate into
\[
b_2^3 - P_1 b_2^2 + (3 + P_2) b_2 - (P_1 + P_3) = 0, \tag{5}
- 3b_2^2 + 2P_1 b_2 + P_1^2 - 4(2 + P_2) \geq 0. \tag{6}
\]
If \( (t_1, t_2, t_3) = F(b_1, b_2, b_3) \) then clearly \( b_2 \) satisfies eqs (5) and (6) and hence is the solution of the system of equations (2).
Conversely if the system of equations (2) has a solution \( y = y_0 \), then set \( b_2 = y_0 \). Therefore \( (t_1 - b_2, t_2 - b_2, t_3 - b_2) \) satisfies equations (4). Now set
\[
b_1 = b_2 + \frac{\bar{P}_1 - \sqrt{\bar{P}_1^2 - 4(2 + \bar{P}_2)}}{2}
\]
and
\[
b_3 = b_2 + \frac{\bar{P}_1 - \sqrt{\bar{P}_1^2 - 4(2 + \bar{P}_2)}}{2}.
\]
This gives \( (t_1, t_2, t_3) = F(b_1, b_2, b_3) \) i.e. \( (t_1, t_2, t_3) \in \text{Range } (F) \). \( \square \)