Auxiliary functions of the Hilbert transform in the study of gravity anomalies

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Abstract. The auxiliary functions, namely amplitude, phase, envelope and instantaneous frequency of Hilbert transform over gravity anomalies of 2-D sphere, vertical fault block and horizontal circular cylinder are studied. The characteristics of these functions are established in locating and identifying the sources. The method is illustrated with a theoretical example in each case and supported by field data of gravity anomaly over Humble dome and spherical model of the Satak-Mansar area, Nagpur District, India.

Keywords. Auxiliary functions; Hilbert transform; instantaneous frequency; amplitude; phase and envelope.

1. Introduction

Hilbert transform is widely used in processing and interpretation of gravity and magnetic anomalies (Nabighian 1974, 1984; Stanley and Green 1976; Green and Stanley 1975; Rao and Ram Babu 1981; Mohan et al 1982; Sundararajan 1982; Sundararajan et al 1983). The characteristics of the amplitude of the analytic signal over magnetic bodies with polygonal cross-section were discussed by Nabighian (1972, 1974). Such studies are valid even in gravity anomalies in addition to spontaneous potential and induced polarization data. In general, the amplitude is useful not only in determining the location of structures and their dimensions etc. but also in the direct interpretation of geophysical data. The Hilbert transform with some auxiliary functions plays a significant role in the resolution of geophysical signals (Ramadass et al 1986).

We present here a simple note on the phase and envelope pertaining to some 2-D gravity anomalies namely, sphere, vertical fault block and horizontal circular cylinder.

If \( g(x) \) is the vertical component of the gravity field which is measurable, its horizontal component \( h(x) \) can be obtained by Hilbert transform. In other words, \( g(x) \) and \( h(x) \) form a Hilbert transform pair. That is,

\[
g(x) \leftrightarrow H \rightarrow h(x)
\]

and

\[
h(x) = \frac{1}{\pi x} * g(x),
\]
where $H$ is the Hilbert transform operator and the asterisk denotes convolution.

Now, the analytic signal which is a complex function can be defined as

$$ f(x) = g(x) + ih(x). $$

It may be noted here that $g(x)$ can also be a horizontal derivative of any order, correspondingly $h(x)$ is the vertical derivative of the same order of $g(x)$.

The amplitude of the analytic signal is defined as

$$ A(x) = \{g(x)\}^2 + \{h(x)\}^2 \right)^{1/2}. $$

(4)

According to Sundararajan (1982) this amplitude can be modified as

$$ A(i \Delta x) = \{g(i \Delta x)\}^2 + \{h(\bar{N} - i + i \Delta x)\}^2 \right)^{1/2}, $$

(5)

where the magnitudes of the Hilbert transforms (4) and (5) remain the same with a phase difference of 180°. Equations (4) and (5) are identical.

The phase which is also useful in direct interpretation is defined as

$$ \text{PH}(x) = \tan^{-1}[-h(x)/g(x)]. $$

(6)

Another auxiliary function which is of some use in delineating the structural characteristics is defined as

$$ \text{EN}(x) = g(x)/\cos[\text{PH}(x)]. $$

(7)

The instantaneous frequency which is the space derivative of envelope function is expressed as

$$ \text{IF}(x) = (d/dx)[\text{PH}(x)]. $$

(8)

In general, the instantaneous frequency is highly oscillating and hence is not included here, although the utility of this frequency function is the same as that of phase and envelope.

The following are the phase and envelope characteristics of the gravity anomalies over sphere and vertical fault block.

2. Theoretical models

2.1 Sphere

The first horizontal derivative of the gravity effect and the corresponding Hilbert transform (vertical derivative) due to a spherical structure (figure 1) are given as

$$ g(x) = \frac{4}{3} \pi R^3 \sigma G \left[ xz/(x^2 + z^2)^{5/2} \right] $$

(9)

and

$$ h(x) = \frac{4}{3} \pi R^3 \sigma G \left[ (x^2 - 2z^2)/(x^2 + z^2)^{5/2} \right], $$

(10)

where $R$ is the radius of the sphere and $Z$ the depth to the centre. The amplitude in this case is given as

$$ A(x) = \frac{4}{3} \pi R^3 \sigma G \left[ (x^4 + 4z^4 - 3x^2 z^2)^{1/2}/(x^2 + z^2)^{5/2} \right]. $$

(11)

At $x = 0$, $A(x)$ reduces to