BOUNDARY VALUES VERSUS DILATATIONS OF HARMONIC MAPPINGS

By

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Abstract. This article is divided into two parts. In the first part, we consider univalent harmonic mappings from the unit disk $U$ onto a Jordan domain $\Omega$ whose dilatation functions $a = f_z/f_{\bar{z}}$ have modulus one on an interval of the unit circle. The boundary values of $f$ depend very strongly on the values of $a(e^{it})$. A complete characterization of the inverse image $f^{-1}(q)$ of a point $q$ on $\partial \Omega$ is given. We then consider the case where the dilatation function $a(z)$ is a finite Blaschke product of degree $N$. It is shown that in this case, $\Omega$ can have at most $N+2$ points of convexity. Finally, we give a complete characterization of simply connected Jordan domains $\Omega$ with the property that there exists a nonparametric minimal surface over $\Omega$ such that the image of its Gaussian map is the upper half-sphere covered exactly once.

1. Introduction

Let $S = (u, v, s)$, $s = g(u, v)$, be a nonparametric surface defined over a simply connected proper subdomain $\Omega$ of the complex plane $C$. Then $S$ is a minimal surface if and only if there exists a univalent complex-valued harmonic mapping $f = u + iv$ from the unit disk $U$ onto $\Omega$ such that $(s_2)^2 = -f_z f_{\bar{z}}$. Note that $f$ can be uniquely expressed in the form $f = h + g$, $g(0) = 0$, where $h$ and $g$ belong to the linear space $H(U)$ of analytic functions on $U$. Without loss of generality, we may assume that $f$ is orientation-preserving (if not, consider the mapping $f(\bar{z})$). It follows that the (second) dilatation function $a = f_z/f_{\bar{z}} = g'/h'$ belongs to $H(U)$ and satisfies $|a| < 1$ on $U$. Historically, the function $i\sqrt{a}$ is called the Weierstrass parameter of the minimal surface and the Gauss map of $S$ is given by the normal vector

$$\vec{N} = \frac{(2 \text{ Im } \sqrt{a}, 2 \text{ Re } \sqrt{a}, 1 - |a|)}{1 + |a|}.$$ 

The study of nonparametric minimal surfaces over $\Omega$ with a given Gauss map leads to the problem of finding univalent harmonic maps from $U$ onto $\Omega$ which are solutions of the elliptic partial differential equation

$$(1.1) \quad f_{\bar{z}}(z) = a(z)f_z(z), \quad z \in U.$$ 

Any nonconstant solution of (1.1) is an orientation-preserving harmonic mapping and any univalent solution is locally quasiconformal on $U$. Observe that the modulus of the dilatation function $a(z)$ may approach one as $z$ tends to the unit
In general, there may exist no univalent solution of (1.1) from $U$ onto $\Omega$. Such, for example, is the case if $\Omega$ is a strictly convex domain and $a(z)$ is a finite Blaschke product [4]. The following result was obtained in [3].

**Theorem A** Let $\Omega$ be a simply connected Jordan domain of $\mathbb{C}$, $w_0 \in \Omega$, and let $a(z) \in H(U)$, $|a| < 1$. Then there exists a univalent solution $f$ of (1.1) such that $f(0) = w_0$, $f'(0) > 0$, $f(U) \subset \Omega$ and, except for a countable set $E$, the nonrestrictive limits $f^*(e^{it}) = \lim_{n \to \infty} f(z)$ exist and are on $\partial \Omega$. For $e^{it} \in E$, the cluster set $C(e^{it}, f)$ of $f$ is a linear line segment in $\Omega$ joining two points of $\partial \Omega$. If $|a(z)| \leq k < 1$ on $U$, then $f(U) = \Omega$ and $f$ extends to a homeomorphism from the closed unit disk $\bar{U}$ onto $\bar{\Omega}$.

In the first part of this article, Section 2, we consider the case in which $a(z)$ admits an analytic extension across an interval $J$ of the unit circle $\partial U$ such that $|a| \equiv 1$ on $J$. The first main result, Theorem 2.2, shows that the boundary values $f^*(e^{it})$ on $J$ depend very strongly on the values of $a(e^{it})$. The second main result of this section is Theorem 2.13, which relates the inverse image $(f^*)^{-1}(q)$ of a boundary point $q \in \partial \Omega$ to the total change of $\arg a(e^{it})$ corresponding to $q$. We present several examples which illustrate these results.

In the second part, Section 3, we study univalent harmonic mappings from the unit disk $U$ onto a Jordan domain $\Omega$ whose (second) dilatation function $a(z)$ is a finite Blaschke product. Our first main result is Theorem 3.3, which states that $\Omega$ has to be a concave regulated domain which contains at most $N + 2$ points of convexity. Moreover, the number of points of convexity plus the number of full resting points is equal to $N + 2$.

Let $\Omega$ be a simply connected Jordan domain of $\mathbb{C}$. In view of Theorem A, we shall define $f^*(e^{it}) = f^*(e^{i(t+\theta)})$ whenever the cluster set $C(e^{it}, f)$ is the line segment in $\Omega$ from $f^*(e^{i(t-\theta)})$ to $f^*(e^{i(t+\theta)})$. In other words, we define $f^*(e^{it})$ as a right-continuous function on $\partial U$.

Let $\Gamma$ be a closed Jordan curve in $\mathbb{C}$. We say that an orientation-preserving mapping $f^*(e^{it})$ from the unit circle $\partial U$ into $\Gamma$ is a **quasihomeomorphism** from $\partial U$ into $\Gamma$ if its image contains at least three noncollinear points of $\Gamma$ and if it is the pointwise limit of a sequence of orientation-preserving homeomorphisms from $\partial U$ onto $\Gamma$. If, in addition, the linear segments from $f^*(e^{it-\theta})$ to $f^*(e^{it+\theta})$ are parts of $\Gamma$, then we call $f^*$ a **quasihomeomorphism** from $\partial U$ onto $\Gamma$. Observe that the functions $f^*(e^{i(t-\theta)})$ and $f^*(e^{i(t+\theta)})$ are always well defined. To see this, let $\phi$ be a conformal (univalent) mapping from $U$ onto the Jordan domain $G$ bounded by $\Gamma$. Then $\arg \phi^{-1} \circ f^*$ is a nondecreasing function on $\partial U$. Since $\phi$ is a homeomorphism on $\partial U$, the assertion follows. Finally, a continuous quasihomeomorphism from $\partial U$ onto $\Gamma$ is called a **weak homeomorphism**. A weak homeomorphism has the property that the inverse image $(f^*)^{-1}(q)$ of each point $q$ on $\partial \Omega$ is a closed interval in $\partial U$. 