TEST FOR PROPERNESS USING RESIDUE CALCULUS

DANIELA ADORNATO - ADELINA FABIANO

We will consider global problems in the ring $K[X_1, \ldots, X_n]$ on the polynomials with coefficients in a subfield $K$ of $\mathbb{C}$.

Let $P = (P_1, \ldots, P_n) : K^n \to K^n$ be a polynomial map such that $(P_1, \ldots, P_n)$ is a quasi-regular sequence generating a proper ideal, the main thing we do is to use the algebraic residues theory (as described in [5]) as a computational tool to give some result to test when a map $(P_1, \ldots, P_n)$ is a proper map by computing a finite number of residue symbols.

1. Introduction.

Let $P = (P_1, \ldots, P_n) : K^n \to K^n$, $K \subseteq \mathbb{C}$, a polynomial map. From the analytic point of view, $P$ is a proper map if

$$\exists \delta > 0, \exists c > 0, \exists h, \|x\| \geq h \Rightarrow \|P(x)\| \geq c\|x\|^\delta$$

where $\delta := \min \left\{ r > 0 : \lim_{\|\xi\| \to \infty} \frac{\|P(\xi)\|}{\|\xi\|^r} > 0 \right\}$ is the Lojasiewicz exponent.

From the algebraic point of view, $P$ is a proper map if $K[X_1, \ldots, X_n]$ is finitely generated module over $K[P_1, \ldots, P_n]$ that is $X_1, \ldots, X_n$ satisfy relations of integral dependency over $K[P_1, \ldots, P_n]$.

Of course, the two different points of view are equivalent.

In [3], the authors used the analytic notion of residue symbols to give some explicit version of the algebraic Nullstellensatz for proper polynomials maps. To do that, they profited from the knowledge of the Lojasiewicz exponent $\delta$ of the map $(P_1, \ldots, P_n)$. The methods were inspired by those used to provide a solution with economic bounds (taking into account the
projective degree) for the algebraic Bezout identity. They proved that when the map is dominant, given \( Q \in \mathbb{C}[X_1, \ldots, X_n] \), there exists a rational function \( R_Q \) in \( n \) variables, with poles along the hypersurface \( \prod_j A_{j0} = 0 \), \( A_{j0} \) being the leading coefficient in the relation of integral dependency of \( X_j \) over \( (P_1, \ldots, P_n) \), such that, for any \( u = (u_1, \ldots, u_n) \) generic, that is outside \( \prod_j A_{j0} = 0 \), one has \( R_Q(u) = \text{Res} \left[ \frac{QdX}{P_1 - u_1, \ldots, P_n - u_n} \right] \).

If the map is proper \( (A_{j0}(u) \equiv 1) \), the residue symbol is a polynomial in \([u_1, \ldots, u_n]\) and it becomes very useful for computations.

Now we would like to look to the proper maps using an algebraic approach. To do that we need to restrict ourselves to the case when the sequence \( (P_1, \ldots, P_n) \) is quasi-regular, which means, since \( K[X_1, \ldots, X_n] \) is Noetherian, the regularity at all common zeroes of \( P_1, \ldots, P_n \).

A sequence \( (P_1, \ldots, P_n) \) generating an ideal \( I \) in \( K[x_1, \ldots, x_n] \) is quasi-regular [6] if and only if whenever one has a relation of the form
\[
\sum_{k \in \mathbb{N}^n, |k| = p} a_k P_1^{k_1} \cdots, P_n^{k_n} \in I^{p+1}
\]
with \( a_k \in K[x_1, \ldots, x_n] \), \( |k| = k_1 + \cdots + k_n \), \( p \in \mathbb{N} \), then all the \( a_k \in I \). In our case, \( K \subseteq \mathbb{C} \) and the analytic definition of residue coincides with the algebraic definition of the residue given by Lipman. Using the last one, we give some result to test the properness just using computations of residue symbols. The key tools are the transformation law and the introduction of additional parameters in order to compute residue symbols.

The advantage of this method is that we do not need some information about the Lojasiewicz exponent and can test the properness of a polynomial map just computing a finite number of residue symbols of the form
\[
\left[ \frac{QdX}{P_1 - u_1, \ldots, P_n - u_n} \right]
\]
and not all of them, namely we need to check if these residue symbols (as functions of the parameters \( u \)) are polynomial functions or not.

2. Residue calculus.

Given \( P_1, \ldots, P_n \) in \( K[X_1, \ldots, X_n] \) such that \( (P_1, \ldots, P_n) \) is a quasi-regular sequence generating the ideal \( (P) = I \) (so that \( K[X_1, \ldots, X_n]/I \)