FELL TOPOLOGY ON THE SPACE
OF FUNCTIONS WITH CLOSED GRAPH

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Let $X$ and $Y$ be $T_1$ topological spaces and $G(X, Y)$ the space of all functions with closed graph. Conditions under which the Fell topology and the weak Fell topology coincide on $G(X, Y)$ are given. Relations between the convergence in the Fell topology $\tau_F$, Kuratowski and continuous convergence are studied too. Characterizations of a topological space $X$ by separation axioms of $(G(X, R), \tau_F)$ and topological properties of $(G(X, R), \tau_F)$ are investigated.

1. Introduction.

For topological spaces $X$ and $Y$ denote by $C(X, Y)$ and $G(X, Y)$ the spaces of all continuous functions from $X$ to $Y$ and of all functions from $X$ to $Y$ with closed graph respectively. As usual $Y^X$ denotes the set of all functions from $X$ to $Y$. For every $f \in Y^X$ let $\Gamma f$ denote the graph of $f$ i.e. $\Gamma f = \{(x, f(x)) : x \in X\}$. By $K(X)$ denote the family of all compact subsets of $X$. In the sequel by $X$ and $Y$ we will always denote $T_1$-spaces.

For a set $A$ in $X \times Y$ denote by $\langle A \rangle = \{f \in Y^X : \Gamma f \cap A = \emptyset\}$. By $\tau_{\langle K \rangle}$ and $\tau_{\langle K_1 \times K_2 \rangle}$ we mean topologies defined by the subbase elements for the open sets $\langle K \rangle$ and $\langle K_1 \times K_2 \rangle$ respectively, where $K$ is compact in $X \times Y$, $K_1$ is compact in $X$, $K_2$ is compact in $Y$ (see [19]).

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By \( \tau_l \) we denote the lower semi-finite graph topology for \( Y^X \), where \( \tau_l \) is given by the subbase elements \([G] = \{ f \in Y^X : \Gamma f \cap G \neq \emptyset \} \) with open sets \( G \subset X \times Y \) (see [19]). Now put \( \tau_F = \tau_{\gamma_k} \vee \tau_l, \tau_{wF} = \tau_{[\gamma_k]} \vee \tau_l \), where the suprema are constructed within the lattice of all topologies for \( Y^X \) (see [22]). \( \tau_F \) is the Fell topology on \( Y^X \) and \( \tau_{wF} \) is the weak Fell topology on \( Y^X \). If \( A \subset X, B \subset Y \) then we will denote \( M(A, B) = \{ f \in Y^X : f(A) \subset B \} \).

The Fell topology on the hyperspace \( C(X) \) of closed (nonempty) subsets of a topological space \( X \) was defined in [8] and then studied in many papers (see [9, 16, 17, 3, 24]). In recent years great interest arose for studying the Fell topology both in spaces of closed sets and in function spaces, stemming for instance from applications to optimization theory (see [1, 2, 4, 12, 6, 15, 23]).

The graph topology \( \tau_{\gamma_k} \) and its weak associated topology \( \gamma_{[\gamma_k]} \) were defined in [19]. In the literature we can find also other hyperspace topologies studied on functions identified with graphs or epigraps (see [1, 22]) as well as multifunctions (see [7]). Some results of our paper were announced in [14].

For undefined notions the reader is referred to a recent monograph [1].

In the first part of our paper we will study on the space of functions relations between the Fell topology and other known topologies as well as relations between the convergence induced from the Fell topology and known convergences (continuous, Kuratowski convergence or topological convergence) on function spaces. In the second part we consider separation axioms and other topological properties, e. g. metrizability, of the space of functions with closed graph equipped with the Fell topology.

2. The Fell topology and other topologies and convergences.

In [19] was shown that if \( X \) and \( Y \) are Hausdorff locally compact spaces then \( \tau_{\gamma_k} = \tau_{[\gamma_k]} \) on \( C(X, Y) \). The following result weakens conditions on spaces \( X, Y \) and also on functions.

**Proposition 2.1.** Let \( X \) and \( Y \) be Hausdorff spaces. Then \( \tau_{\gamma_k} = \tau_{[\gamma_k]} \) on \( G(X, Y) \).