We study the notion of \( \psi \)-density point and \( \psi \)-density topologies for non-decreasing continuous functions \( \psi \). Also basic properties of \( \psi \)-approximately continuous functions are considered.

0. Introduction.

The purpose of this paper is to study the notion of a \( \psi \)-density point and \( \psi \)-density topology, generated by it analogously to the classical density topology on the real line. The idea arises from the note by S. J. Taylor [3]. To keep the analogy more close, we shall use only symmetric intervals (see Thms. 0.1 and 0.2 below). The paper consists of three sections. In the first one we introduce the notions of a \( \psi \)-density point as well as of the \( \psi \)-density topology and consider its properties. In the second, there are studied relationships between topologies generated by different functions. In the last section we consider \( \psi \)-approximately continuous functions and prove the theorem analogous to the Denjoy Theorem concerning measurable functions. We shall use the following notations. Let \( \mathbb{N} \) denote the set of positive integers, \( \mathbb{Q}(\mathbb{Q}_+) \)-the set of rational (positive rational) numbers, \( \mathbb{R}(\mathbb{R}_+) \) - the set of real (positive real) numbers, \( \mathcal{S} \) - the \( \sigma \)-algebra of Lebesgue measurable sets and \( m \) - the Lebesgue measure. If \( A \subset \mathbb{R} \), then \( A' = \mathbb{R}\setminus A \). We shall say that two sets \( A \) and \( B \) are equivalent \( (A \sim B) \) if \( m(A\Delta B) = 0 \), where \( A\Delta B \) is the symmetric difference of \( A \) and \( B \). We say that \( x \in \mathbb{R} \) is a density
point of a set $A \in S$ if and only if
$$\lim_{h \to 0^+} \frac{m(A \cap [x - h, x + h])}{2h} = 1$$
or, which is more convenient for us,
$$\lim_{h \to 0^+} \frac{m(A' \cap [x - h, x + h])}{2h} = 0.$$

Let $\phi(A) = \{x \in \mathbb{R} : x \text{ is a density point of } A\}$ for $A \in S$. Then the family $d = \{A \in S : A \subset d(A)\}$ is a topology on a real line, called the density topology (see [1], Th.22.5). We shall denote by $\mathcal{O}$ the Euclidean topology on the real line. Let $C$ denote the family of all continuous nondecreasing functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x \to 0^+} \psi(x) = 0$. S. J. Taylor considered in [3] the possibility of the improvement of the Lebesgue Density Theorem, which says that if $E \in S$, then the equality

\[(0.1) \quad \lim_{m(I) \to 0} \frac{m(E' \cap I)}{m(I)} = 0\]

holds for all points $x \in E$ except for a set of measure zero. Ylor proved that if we consider a stronger condition

\[(0.2) \quad \lim_{m(I) \to 0} \frac{m(E' \cap I)}{m(I)\psi(m(I))} = 0\]

where $\psi \in C$, then the result of Lebesgue is the 'best possible' in the sense that it cannot be improved uniformly for all measurable sets, but for each set $E \in S$, there exists a function $\psi \in C$ such that (0.2) holds for almost all $x \in E$ ([3], Th.3). In our paper we shall use only intervals $I$ with centre at $x$. Then, instead of (0.2), we shall consider the condition

\[(0.3) \quad \lim_{h \to 0^+} \frac{m(E' \cap [x - h, x + h])}{2h\psi(2h)} = 0.\]

Clearly, (0.2)$\Rightarrow$(0.3). However, the inverse condition does not hold.

**Theorem 0.1.** There exist a measurable set $A$ and a function $\psi \in C$, such that, for a point $x = 0$ condition (0.3) is fulfilled, but (0.2) does not hold.