LOTOTSKY-SCHNABL OPERATORS
ON THE UNIT INTERVAL*

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Let \( \lambda : [0, 1] \rightarrow [0, 1] \) be a continuous function. For \( n \in \mathbb{N} \), the \( n \)th Lototsky-Schnabl operator on \([0, 1]\), \( L_{n, \lambda} \), associated with \( \lambda \) is defined by

\[
L_{n, \lambda}(f, x) := \int_0^1 \cdots \int_0^1 f \left( \frac{x_1 + \cdots + x_n}{n} \right) d\nu_x(x_1) \cdots d\nu_x(x_n),
\]

where \( x \in [0, 1] \), \( f \in M := \) the set of all real bounded functions on \([0, 1]\), and the probability measure \( \nu_x \) is the convex linear combination

\[
\nu_x := \lambda(x) \varepsilon_x + (1 - \lambda(x)) \delta_x,
\]

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\( \epsilon_x \) (resp. \( \delta_x \)) being the Bernoulli distribution with parameter \( x \) (resp. the Dirac measure, or unit mass, at \( x \)). Observe that, \( \nu_x \) being discrete, no measurability condition on \( f \) is required.

It is immediate that \( L_{n,\lambda} f \) interpolates \( f \) at 0 and 1. We also have, for \( x \in [0, 1] \),

\[
L_{n,\lambda}(f, x) = \begin{cases} 
  f(x), & \text{if } \lambda(x) = 0, \\
  B_n(f, x), & \text{if } \lambda(x) = 1,
\end{cases}
\]

where \( B_n \) is the classical Bernstein operator

\[
B_n(f, x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1 - x)^{n-k}.
\]

Therefore, \( L_{n,\lambda} \) coincides with \( B_n \) or the identity operator according to \( \lambda \equiv 1 \) or \( \lambda \equiv 0 \).

On the other hand, we can write (see [5])

\[
(3) \quad L_{n,\lambda}(f, x) = \sum_{r=0}^{n} \sum_{h=0}^{r} f \left( \frac{h}{n} + \left( 1 - \frac{r}{n} \right) x \right) p_{n,\lambda}(r, h, x),
\]

where

\[
(4) \quad p_{n,\lambda}(r, h, x) := \binom{n}{r} \binom{r}{h} [\lambda(x)]^r [1 - \lambda(x)]^{n-r} x^h (1 - x)^{r-h},
\]

and

\[
(5) \quad L_{n,\lambda}(f, x) = \sum_{r=0}^{n} \binom{n}{r} [\lambda(x)]^r [1 - \lambda(x)]^{n-r} B_r(f_{x,\lambda}, x),
\]

where

\[
(6) \quad f_{x,a}(z) := f(ax + (1 - a)x), \ a, x, z \in [0, 1],
\]

and (here and hereafter) it is understood that \( B_0 \) is the identity operator.

The operators \( L_{n,\lambda} \) are one-dimensional versions of Lototsky-Schnabl operators associated with Altomare projections, which have been introduced by Altomare in connection with the investigation of the qualitative properties of the solutions to certain classes of degenerate diffusion equations [5-8] (see also [9]). In this direction, the following result is particularly significant (cf. [9; Corollary 6.3.6]).