INVERSIVE PLANES WITH CIRCLES DETERMINED
BY TANGENTS

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In an inversive plane \( J \), one derived affine plane is such that every circle is uniquely determined by straight line tangents. The validity of 4-point Pascal conditions on one of the circles then plays a role in the coordinatization of \( J \).

The purpose of this paper is the study of inversive planes \( J \) in whose derived affine plane every circle is uniquely determined by two straight line tangents from each of two fixed pencils of parallels. These pencils are then used to set up a coordinate system in the derived plane of \( J \). After postulating transitivity of the derived plane under all dilatations, the coordinate structure becomes a skewfield. The plane \( J \) is then of Hering type III2. Now the question arises what additional conditions, as weak as possible, to impose for making \( J \) miquelian. Of course, Miquel’s condition, or Pascal’s full condition on one of the circles would do, but it turns out that it suffices to postulate 4-point Pascal conditions on one of the circles, even with the restriction that 3 of the 4 points on the circle are fixed. It is well known [5] that the 5-point Pascal condition is equivalent to the full 6-point condition. However, not much has been known on the 4-point Pascal condition, which now turns out to be able to play an important role in the coordinatization of an inversive plane over a field, once the derived affine plane is desarguesian. The paper concludes with a remark showing that the 4-point Pascal condition used leads to a construction which makes the circle into a conic in the sense of Steiner.

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1. Definitions.

An inversive planes $J$ consists of points and subsets of points called circles such that:

I) 3 distinct points determine a unique circle.

II) Given a circle $c$, a point $P$ on $c$, and a point $Q$ not on $c$. Then there is a unique circle $d$ containing $Q$ and such that $c \cap d = \{P\}$.

III) Each circle contains a point. There exist 4 points not on the same circle.

The following definitions and notations will be used, most of them in accordance with [4]:

Circles having just one point in common are called tangent. The set of all circles tangent at a point $P$ is called a pencil, $P$ its carrier. The pencil with carrier $P$ containing a circle $c$ is called $Pc$. The set of all circles containing distinct points $P$ and $Q$ is called the bundle $(P, Q)$. The circle through $A, B, C$ is called $ABC$. The «derived» plane containing all points of $J$ except $P$, and all circles through $P$, is called $J_F$.

In addition, we will consider the following axioms.

IV) $J$ contains a point $X$ such that in $J_X$ no diagonals of any parallelogram are parallel.

V') For every pencil $Pa$ and for every circle $c$ not through $P$, there are exactly 2 circles in $Pa$ that are tangent to $c$.

The following is a weaker form of V'.

V) In $J$ there is a circle $k$ not containing $X$ such that:

(i) there is a pencil $\pi_1$ with carrier $X$ containing exactly two circles $a_1$ and $a_2$ tangent to $k$;

(ii) if $a_1 \cap a_2 = : P_1, a_2 \cap k = : P_2$, then the pencil $X(XP_1P_2) = : \pi_2$ contains exactly two circles $b_1$ and $b_2$ tangent to $k$;

(iii) the pencil $P_1a_1$ contains only the two circles $k$ and $a_1$ tangent to $a_2$;

(iv) every circle of $J$ not containing $X$ has exactly two circles from $\pi_1$ tangent to it.

Obviously V' implies V. Every plane satisfying I through IV and V' is an (F)-plane [2]. Planes satisfying Axiomes I through V will be called $T$-planes. They are not necessarily (F)-planes.