INVERSIVE PLANES WITH CIRCLES DETERMINED BY TANGENTS

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In an inversive plane $J$, one derived affine plane is such that every circle is uniquely determined by straight line tangents. The validity of 4-point Pascal conditions on one of the circles then plays a role in the coordinatization of $J$.

The purpose of this paper is the study of inversive planes $J$ in whose derived affine plane every circle is uniquely determined by two straight line tangents from each of two fixed pencils of parallels. These pencils are then used to set up a coordinate system in the derived plane of $J$. After postulating transitivity of the derived plane under all dilatations, the coordinate structure becomes a skewfield. The plane $J$ is then of Hering type III2. Now the question arises what additional conditions, as weak as possible, to impose for making $J$ miquelian. Of course, Miquel's condition, or Pascal's full condition on one of the circles would do, but it turns out that it suffices to postulate 4-point Pascal conditions on one of the circles, even with the restriction that 3 of the 4 points on the circle are fixed. It is well known [5] that the 5-point Pascal condition is equivalent to the full 6-point condition. However, not much has been known on the 4-point Pascal condition, which now turns out to be able to play an important role in the coordinatization of an inversive plane over a field, once the derived affine plane is desarguesian. The paper concludes with a remark showing that the 4-point Pascal condition used leads to a construction which makes the circle into a conic in the sense of Steiner.

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1. Definitions.

An inversive planes \( J \) consists of points and subsets of points called circles such that:

I) 3 distinct points determine a unique circle.

II) Given a circle \( c \), a point \( P \) on \( c \), and a point \( Q \) not on \( c \). Then there is a unique circle \( d \) containing \( Q \) and such that \( c \cap d = \{ P \} \).

III) Each circle contains a point. There exist 4 points not on the same circle.

The following definitions and notations will be used, most of them in accordance with [4]:

Circles having just one point in common are called tangent. The set of all circles tangent at a point \( P \) is called a pencil, \( P \) its carrier. The pencil with carrier \( P \) containing a circle \( c \) is called \( Pc \). The set of all circles containing distinct points \( P \) and \( Q \) is called the bundle \( (P, Q) \). The circle through \( A, B, C \) is called \( ABC \). The « derived » plane containing all points of \( J \) except \( P \), and all circles through \( P \), is called \( J_P \).

In addition, we will consider the following axioms.

IV) \( J \) contains a point \( X \) such that in \( J_X \) no diagonals of any parallelogram are parallel.

V') For every pencil \( Pa \) and for every circle \( c \) not through \( P \), there are exactly 2 circles in \( Pa \) that are tangent to \( c \).

The following is a weaker form of V'.

V) In \( J \) there is a circle \( k \) not containing \( X \) such that:

(i) there is a pencil \( \pi_1 \) with carrier \( X \) containing exactly two circles \( a_1 \) and \( a_2 \) tangent to \( k \);

(ii) if \( a_1 \cap a_2 = : P_1 \), \( a_2 \cap k = : P_2 \), then the pencil \( X(XP_1P_2) = : \pi_2 \) contains exactly two circles \( b_1 \) and \( b_2 \) tangent to \( k \);

(iii) the pencil \( P_1a_1 \) contains only the two circles \( k \) and \( a_1 \) tangent to \( a_2 \);

(iv) every circle of \( J \) not containing \( X \) has exactly two circles from \( \pi_1 \) tangent to it.

Obviously V' implies V. Every plane satisfying I through IV and V' is an \((F)\)-plane [2]. Planes satisfying Axiomes I through V will be called \( T \)-planes. They are not necessarily \((F)\)-planes.