WEAK TYPE INEQUALITIES OF MAXIMAL HANKEL CONVOLUTION OPERATORS(*)

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In this paper we characterize weak type (1,1) inequalities for Hankel convolution operators by means of discrete methods.

1. Introduction.


\[ h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x) d\gamma(x), \quad y \in (0, \infty), \]

where \( d\gamma(x) = x^{\mu+1} dx \) and \( J_\mu \), as usual represents the Bessel function of the first kind and order \( \mu \). Throughout this paper we will assume that \( \mu \) is greater than \(-1/2\). We consider the space \( L_p(\gamma) \), \( 1 \leq p < \infty \), that consists of all those measurable functions \( f \) on \( (0, \infty) \) such that

\[ \|f\|_p = \left( \int_0^\infty |f(x)|^p d\gamma(x) \right)^{1/p} < \infty. \]

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It is well-known that \( L_p(y) \) endowed with the norm \( \| \cdot \|_p \) is a Banach space.

The convolution operation for the \( h_\mu \)-transformation is defined as follows (see [3], [6], [8], [9], [11], [12] and [13], amongst others). Let \( f \) and \( g \) be in \( L_1(y) \). The Hankel convolution \( f \# g \) of \( f \) and \( g \) is defined by

\[
(f \# g)(x) = \int_0^\infty (\tau_x f)(z)g(z)\,dy(z), \quad \text{a.e. } x \in (0, \infty),
\]

where the Hankel translated \( \tau_x f \) of \( f \) by \( x \in (0, \infty) \) is

\[
(\tau_x f)(z) = \int_0^\infty f(y)dW_{x,z}(t), \quad \text{a.e. } z \in (0, \infty),
\]

and being, for every \( x, z \in (0, \infty) \), \( W_{x,z}(y) \) the probability measure defined by

\[
dW_{x,z}(y) = a_\mu \frac{\Delta(x, \gamma, z)^{2\mu-1}}{(xyz)^{2\mu}}\,dy(y),
\]

where \( a_\mu = 2^{1-2\mu}\Gamma(\mu + 1)\Gamma(\mu + 1/2)^{-1}\pi^{-1/2} \) and

\[
\Delta(x, y, z) = \begin{cases} 
[((x + z)^2 - y^2)(y^2 - (x - z)^2)]^{1/2}, & \text{when } |x - z| < y < x + z, x, z \in (0, \infty) \\
0, & \text{otherwise}.
\end{cases}
\]

By defining on \( L_1(y) \) the Hankel convolution, \( L_1(y) \) becomes a commutative Banach algebra ([12]).

Let \( (k_j)_{j=1}^\infty \) be a sequence in \( L_1(y) \). We define the maximal Hankel convolution operator associated to \( (k_j)_{j=1}^\infty \) by

\[
K^* f(x) = \sup_{j \in \mathbb{N}} |(k_j \# f)(x)|, \quad f \in L_1(y) \text{ and } x \in (0, \infty).
\]

In this paper, inspired in results of [1], [7] and [10], we characterize (Theorem 2.1) the weak type (1,1) inequality for the operator \( K^* \) by means of the weak type (1,1) inequalities for the operators defined by

\[
\sup_{j \in \mathbb{N}} |K_{x_1, \ldots, x_n, k_j}(x)| \quad \text{where}
\]

\[
(K_{x_1, \ldots, x_n} f)(x) = \left| \sum_{i=1}^n (\tau_{x_i} f)(x) \right|, \quad x \in (0, \infty) \text{ and } f \in L_1(y),
\]