FINITE UNIONS OF SHORE SETS

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A dendroid is an arcwise connected hereditarily unicoherent continuum. A shore set in a dendroid $X$ is a subset $A$ of $X$ such that, for each $\epsilon > 0$, there exists a subdendroid $B$ of $X$ such that the Hausdorff distance from $B$ to $X$ is less than $\epsilon$ and $B \cap A = \emptyset$.

Answering a question by I. Puga, in this paper we prove that the finite union of pairwise disjoint shore subdendroids of a dendroid $X$ is a shore set. We also show that the hypothesis that the shore subdendroids are disjoint is necessary. It is still unknown if the union of two closed disjoint shore subsets of a dendroid $X$ is also a shore set.

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Introduction.

A continuum is a compact connected metric space. Given a continuum $X$, we denote by $2^X$ the hyperspace of all closed nonempty subsets of $X$, with the Hausdorff metric $H$. We denote by $C(X)$ the subspace of $2^X$ consisting of all the subcontinua of $X$.

A dendroid is an arcwise connected hereditarily unicoherent (i.e., $A \cap B$ is connected for every $A, B \in C(X)$) continuum. Given a dendroid $X$ and
points \(a, b \in X\), we denote by \(ab\) the unique arc in \(X\) joining \(a\) and \(b\), if \(a \neq b\) and \(ab = \{a\}\) if \(a = b\). A dendrite is a locally connected dendroid. A dendroid \(X\) is said to be smooth provided that there exists a point \(p \in X\) such that, for each sequence \(\{x_n\}_{n=1}^\infty\), converging to a point \(x \in X\), \(px_n \to px\).

A subset \(A\) of a dendroid \(X\) is said to be a shore set provided that, for each \(\epsilon > 0\), there exists a subcontinuum \(B\) of \(X\) such that \(B \cap A = \emptyset\) and \(H(B, X) < \epsilon\). A point \(p\) in \(X\) is called a shore point provided that \(\{p\}\) is a shore set.

Shore sets were introduced in [4] where they were used to study some regularities of the neighborhoods of the element \(X\) in the hyperspace \(C(X)\), for a dendroid \(X\). Later, V. Neumann-Lara and I. Puga-Espinosa studied relations between shore points and noncut points in dendroids in [6], and they used shore points to characterize dendrites ([5, Theorem 2.1]).

In [4, Theorem 1.6], Montejano-Peimbert and Puga-Espinosa proved that every finite set of shore points in a smooth dendroid \(X\) is a shore set. In the Continuum Theory Seminar of the National University of Mexico, I. Puga-Espinosa asked if the hypothesis of smoothness can be removed from this theorem. In this paper we answer Puga-Espinosa’s question by proving the following result.

**THEOREM.** If \(X\) is a dendroid and \(A_1, A_2, \ldots, A_m\) are pairwise disjoint shore subcontinua of \(X\), then \(A_1 \cup A_2 \cup \ldots \cup A_m\) is a shore set.

We also show that pairwise disjointness is necessary in this theorem by giving an example of a dendroid \(Z\) such that there are two (non-disjoint) shore subcontinua \(A\) and \(B\) of \(Z\) such that \(A \cup B\) is not a shore set of \(Z\). The natural question: Is the union of two disjoint closed shore subset of a dendroid \(Z\) a shore set? is still open.

In general, the union of two shore sets is not necessary a shore set. In order to see this, let \(X\) be the cone over the Cantor set \(C\), take two disjoint dense subsets \(A, B\) of \(C\) such that \(C = A \cup B\). Thus, \(A \times \left\{ \frac{1}{2} \right\} \subset X\) and