ON THE TREES ASSOCIATED TO PLANE CURVE SINGULARITIES AND IDEALS

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Si comparano diverse nozioni d'equisingolarità per le curve tracciate sopra una superficie complessa liscia, e per gli ideali di dimensione finita dell'anello locale d'uno dei suoi punti. Si vede che, in generale, questi nozioni sono diverse, ma (per le curve) queste coincidono se esse fossero membri della stessa famiglia continua.

Introduction.

A possible approach to study the complexity of singularity of a reduced plane curve, or that of an ideal in a regular two dimensional local ring (primary to the maximal ideal) is to consider its associated tree of "infinitely near points". This is close to the spirit of the original investigations of Federigo Enriques on the subject. Here, each vertex of the tree corresponds to a point infinitely near to the singularity (or the support of the ideal), two vertices being joined by an edge if one is obtained from the other by means of a single quadratic transformation. The tree itself does not say much about the singularity (or ideal), but one may enrich it by assigning integral "weights" to the vertices. For instance, one may take the multiplicity (or the order) of the appropriate proper transform, or some other numbers that
essentially describe the proximities between points, or a combination thereof, etc. One obtains different notions of equivalence, or equisingularity, for curves (or ideals) by requiring that the various weighted trees that appear be isomorphic (by means of an isomorphism respecting the weights). However, the situation is different when we are dealing with curves which are members of the same family. In this case, several different natural notions of equivalence agree.

More precisely, in this note we discuss, along the lines just mentioned, what we call the notions of $R$- and $C$-equivalence of ideals (see section 3). The concept of $R$-equivalence was studied by Risler in [4] and by this author in [3]. The notion of $C$-equivalence was suggested to me by E. Casas. In the same section, I discuss the notions of $Z$- and $M$-equivalence of reduced curves ($Z$-equivalence being the same as the well-known concept of Zariski-equivalence, see [5]).

In section 3 (Theorem 2) it is checked that given a family of curves $\{C_t\}$, $t$ in a connected parameter space $T$, then the different fibers are $Z$-equivalent to each other if and only if they are $M$-equivalent (although in general only one implication is valid). Concerning families of ideals, to require that all members be $C$-equivalent is strictly stronger than requiring $R$-equivalence, and the former condition (but not the second) is equivalent to requiring (essentially) that, "in general", an element of the defining ideal determines an equisingular family of plane curve singularities, contrary to the assertion of Proposition 3 in [4]. See Theorem 1 for the precise statement.

In the last section we indicate how these theorems allow us to improve the results of an earlier paper of this author.

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To simplify the presentation, we work throughout in the analytic context, over the field $\mathbb{C}$ of complex numbers.