ON $\sigma$-SATURATIONS OF A BOOLEAN ALGEBRA

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In the study of classes of extensions of a Boolean Algebra, two kinds of problems arise. The first one concerns the existence of a minimal element, with respect to a suitably defined ordering, in the given extension class; the second problem, which is often related to the one just mentioned above, is whether two different extensions belonging to the same class are isomorphic or not. It's well known that, in the case of completions of a given algebra, the answer is positive to both questions. On the other hand, if we consider the class of $(J, M, m)$ extensions [2], [5] introduced by Kerstan, we have that a given algebra $\mathcal{B}$ can admit non-isomorphic extensions of this kind. With regard to this subject, if $K$ is the class of $(J, M, m)$ extensions with a suitable (quasi)-ordering, it's still open the problem whether the $m$-completions are the smallest members in the given ordering.

In the present paper we are going to introduce the class of $\sigma$-saturations of a Boolean Algebra. However the definitions related to the ordering in this class are formally very alike those concerning the completion, we obtain in this case very different results. As a matter of fact, there is no minimal $\sigma$-saturation for all Boolean Algebras (th. 1.1). Such a negative outcome allows a tool for the classification of Boolean Algebras. They can be divided into three classes: $C_1$ = class of $\sigma$-sature algebras; $C_2$ = class of $\sigma$-saturable algebras
(namely admitting a minimal $\sigma$-saturation); $\mathcal{C}_1$ = class of non $\sigma$-saturable algebras. We have that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and, if $\mathcal{C}$ is the class of complete algebras whose cardinality is less than the first (Ulam)-measurable cardinal, then $\mathcal{C} \subseteq \mathcal{C}_1$ (theorem 2.1). Furthermore, every $\sigma$-sature algebra is $\sigma$-perfect, while the converse is false as we show in the Example I.

1. We take for granted the basic concepts concerning Boolean Algebras (abbr. « algebra »); for any of them see [1]. We recall that, if $\mathcal{B}'$, $\mathcal{B}''$ are algebras and $\mathcal{B}' \subseteq \mathcal{B}''$, we say that $\mathcal{B}'$ generates $\mathcal{B}''$ whenever each element of $\mathcal{B}''$ is the supremum of elements of $\mathcal{B}'$. A homomorphism between algebras is said to be complete if it preserves all suprema which happen to exist. A maximal ideal $\mathcal{I}$ in an algebra is $\sigma$-complete if given any sequence $\{t_n\}_{n<\omega}$ of elements, $t_n \in \mathcal{I}$, we get $\bigvee t_n \in \mathcal{I}$, $n<\omega$. An ideal $\mathcal{I}$ is called principal when, $\forall \{t_n\}_{n \in \mathcal{I}}$, $t_n \in \mathcal{I}$, $\bigvee t_n \in \mathcal{I}$. An algebra $\mathcal{B}$ is $\sigma$-sature if every $\sigma$-complete maximal ideal is principal. If $\mathcal{B}$ is a given algebra, we say that $(\mathcal{C}, j)$ is a $\sigma$-saturation of $\mathcal{B}$ if $\mathcal{C}$ is a $\sigma$-sature algebra, $j: \mathcal{B} \rightarrow \mathcal{C}$ a complete monomorphism and moreover $j(\mathcal{B})$ generates $\mathcal{C}$. A $\sigma$-saturation $(\mathcal{S}, i)$ is called minimal if for any other $\sigma$-saturation $(\mathcal{C}, j)$ there is a complete monomorphism $h: \mathcal{S} \rightarrow \mathcal{C}$ such that $j = h \circ i$.

Let's point out that the definitions of the above coincide with those related to the completion if we substitute the expression « complete algebra » for « $\sigma$-sature algebra ». As to the completion of a Boolean Algebra, we recall that: a) two completions $\mathcal{C}$ and $\mathcal{C}'$ of an algebra $\mathcal{B}$ are always isomorphic; b) every algebra admits a minimal completion, which is unique up to isomorphisms.

**Theorem 1.1.** Let $S$ be a set, with $|S| > \omega$, and $\mathcal{C}_S$ the countable-cocountable algebra on $S$, namely the $\sigma$-algebra generated by all subsets of cardinality at most countable. Then $\mathcal{C}_S$ does not admit a minimal $\sigma$-saturation.

**Proof.** Let's first remark that $\mathcal{C}_S$ is obviously non sature since the ideal of finite and countable sets is not principal. Let's define the following preorder on $2^\omega$: given $f = \{a_n\}$, $g = \{b_n\}$, $n<\omega$, we pose $f \leq g$ if there is $n_0 \geq 0$ such that $a_n = b_{n_0 + n}$; $\forall n<\omega$. Let's define now, by transfinite induction, a set of $\omega_1$ pairwise incomparable sequences. Notice that, for any fixed map $f \in 2^\omega$, there