GLOBAL EXISTENCE 
IN THE FUTURE AND BOUNDEDNESS 
OF SUBMANIFOLDS OF SOLUTIONS 
OF A SCALAR COMPARISON EQUATION

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Sufficient conditions for the global existence in the future, and for the boundedness, of solutions of the scalar differential comparison equation $\dot{u} = a(t)f(u) + b(t)g(u)$ are provided.

1. Introduction.

Let us consider the differential system

$$\dot{x} = \phi(t, x),$$

(1.1)

where $\phi : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is an appropriately smooth function, and for convenience we write $\mathbb{R}^+ = [0, \infty)$.

Suppose that there exists a Liapunov function $V : \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^+$ whose time-derivative, calculated along the solutions of (1.1), satisfies the following differential inequality

This research has been partly supported by the Italian Ministero dell’Università e della Ricerca Scientifica e Tecnologica, fondi 40%.
\( \dot{V}_{(1.1)} \leq p(t, V(t, x)) \text{ on } \mathbb{R}^+ \times \mathbb{R}^n, \)

where \( p : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R} \) is a continuous function.

As is well known, the global existence in the future, or the boundedness, of all solutions of the comparison equation

\( \dot{u} = p(t, u) \)

implies (under appropriate hypotheses) the global existence in the future or the boundedness, of all solutions of the system (1.1) (see e.g. [5-9]).

An analogous property holds when there is a submanifold of solutions of the comparison equation (1.3) which exist globally in the future, or are bounded. An example of this has been given in [10]. Another example will be provided in Section 4 of this paper, in which Corollary 2 of [2] will be generalized by choosing the Euclidean norm of the vector \( x \) as a Liapunov function.

This justifies the search for sufficient conditions for the global existence in the future, and for the boundedness, of solutions of equation (1.3).

Here we consider the scalar comparison equations of the type

\( \dot{u} = a(t)f(u) + b(t)g(u), \)

where the functions \( a : \mathbb{R}^+ \to \mathbb{R} \) and \( b, f, g : \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous and satisfy the conditions

\( f(u) > 0 \text{ on } (0, \infty), \text{ and } \int_0^\infty \frac{du}{f(u)} = \infty. \)

\( \exists T \geq 0 : a(t)f(0) + b(t)g(0) \geq 0 \text{ on } [T, \infty). \)

By means of the lemmas in Section 2, we then find sufficient conditions for the global existence in the future, and for the