GENERALIZATION OF P-HYPERGROUPS

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Si definiscono e si studiano larghe classi di ipergruppi derivanti da comuni semi-gruppi.

1. Introduction.

A hypergroup, in the sense of Marty (1934), is a set $H$ equipped with an associative hyperoperation $\cdot : H \times H \rightarrow P(H)$ which satisfies the property that $xH = Hx = H$, for all $x \in H$. In this paper we consider hypergroups constructed from ordinary semigroups which generalized the notion of $P$-hypergroups introduced in [6]. In particular we study the cyclicity and reversibility properties of these structures.

For a preliminary report see author’s Generalization of $P$-hypergroups, Abstracts, Amer, Math. Soc. 37, 6 (2) p. 234 (1985).

2. Definitions.

Let $(G, \cdot)$ be a semigroup and $P \subset G$, $P \neq \emptyset$. We shall call $P$-\textit{semihypergroup} the hyperstructure $(G, ^\ast)$ where the $P$-\textit{hyperoperation} $^\ast$ is defined as follows

$$^\ast : G \times G \rightarrow P(G) : (x, y) \mapsto xP^\ast y$$
In fact \( (G, \cdot) \) is a semihypergroup because
\[
(x \cdot y) \cdot P z = z P y = y \cdot (y \cdot P z), \quad \forall x, y, z \in G
\]

Call \( (G, \cdot) \) a \( P \)-hypergroup if it is a hypergroup in the sense of Marty, i.e. \( (G, \cdot) \) is a \( P \)-semihypergroup and the reproduction axiom is valid:
\[
x \cdot P G = G \cdot P x = G, \quad \forall x \in G
\]

If \( P \) is a non empty subset of a group \( G \), then \( (G, \cdot) \) is a \( P \)-hypergroup because we have
\[
x \cdot P G = \bigcup_{g \in G} x \cdot P g = \bigcup_{g \in G} x \cdot g P = x P G = G
\]

and similarily \( G \cdot P x = G \). We shall mainly consider this case. A subset \( H \) of \( G \) will be called \( P \)-subhypergroup of \( (G, \cdot) \) if \( P \subseteq H \subseteq G \) and \( (H, \cdot) \) is a hypergroup. In this paper we shall always assume \( P \neq \emptyset \).

The above definitions are generalizations of the ones in [6]. We simply note that in [6] the set \( \{e\} \cup P \) stands instead of \( P \) above.

Now let us exclude the degenerate cases of groups and total hypergroups \( (G, \circ) \) i.e. \( x \circ y = G, \forall x, y \in G \). Then in the paper [3] it is proved that all cyclic and complete hypergroups [1] of rank less than 6 are isomorphic to those that are given there with multiplication tables. From those hypergroups only one is \( P \)-hypergroup: specifically the \( H_3 \) in Th. (4) which is a hypergroup with four elements and it can be obtained from the cyclic group \( G = \{a, a^2, a^3, a^4 = e\} \) with \( P = \{e, a^2\} \). We can see this setting in [3] \( x = e, y = a^2, z = a, t = a^3 \).

3. Some hyperhomomorphisms.

**PROPOSITION** Let \( (G_1, \cdot) \), \( (G_2, \circ) \) be two groups, \( f \in \text{Hom}(G_1, G_2) \) and \( P \subseteq G_1 \). Then the homomorphism \( f \) is a strong hyperhomomorphism between the \( P \)-hypergroups \( (G_1, \cdot) \) and \( (G_2, f(P)) \).

**Proof.** We have \( \forall x, y \in G_1 \)
\[
f(x \cdot y) = f(x P y) = f(x) \circ f(P) \circ f(y) = f(x)f(P)f(y) \quad Q.E.D.
\]

In this proposition if \( f \) is an isomorphism then it will also be a hyperisomorphism.