HOMOLOGY, HOMOMORPHISMS AND PRESENTATIONS
OF COMMUTATIVE ALGEBRAS

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We obtain sufficient conditions for that a subalgebra of a commutative algebra be a polynomial algebra.

1. Introduction.

The purpose of this paper is the use of Andre's homology to obtain for Commutative Algebra similar results to the obtained by Stammbach [4] and Knus [2] in Group Theory and Associative Algebra respectively.

All rings and algebras will be commutative and with unit, and a ring homomorphism will map unit into unit.

The paper is divided into two parts. In the first we obtain sufficient conditions for that the subalgebra generated by a set of elements of a supplemented algebra be a polynomial algebra. In the second part we consider algebras over a principal ideal domain (PID) and we define the concept of deficiency of a finitely presented algebra. We obtain an upper bound for the deficiency in terms of the first homology module. We also generalize some results of the first part.

2. Polynomial subalgebra theorems.

We start with an useful lemma to obtain an analogous to the theorem
by Stallings-Stammbach ([4], theorem IV.1.1) in the homology of commutative algebras.

**Lemma 1.1.** a) Let $A$ be a ring, $B \twoheadrightarrow B/J$ a surjective homomorphism of algebras over $A$ and $W$ a $B/J$-module. Then there exists an exact and natural sequence

\[
H_1(A, B, W) \longrightarrow H_1(A, B/J, W) \longrightarrow J/\mathfrak{p} \otimes_{B/J} W \longrightarrow H_0(A, B, W) \longrightarrow H_0(A, B/J, W)
\]

b) Let $I$ be another ideal of $B$ such that $I \subseteq J$. Then there exists an exact and natural sequence

\[
H_1(A, B, B/I) \longrightarrow H_1(A, B/J, B/I) \longrightarrow I/J \rightarrow H_0(A, B, B/I) \longrightarrow H_0(B, B/J, B/I)
\]

**Proof.** a) It is a consequence of Jacobi-Zariski exact sequence ([1], theorem 5.1) and of the following two facts: $H_0(B, B/J, W) = 0$ ([1], lemma 4.60) and $H_1(B, B/J, W) = J/\mathfrak{p} \otimes_{B/J} W$ ([1], proposition 6.1).

b) Since $J/\mathfrak{p} \otimes_{B/J} B/I \cong J/IJ$, it is a consequence of a).

**Theorem 1.2.** Let $A$ be a ring, $B, C$ algebras over $A$, $I$ an ideal of $B$, $J$ an ideal of $C$ and $f: B \rightarrow C$ a homomorphism of algebras which induces an isomorphism $B/I \cong C/J$. If $f$ induces an isomorphism $I/\mathfrak{p} \cong J/\mathfrak{p}$ and an epimorphism

\[
H_1(A, B, B/I) \twoheadrightarrow H_1(A, C, C/J)
\]

then $f$ induces isomorphisms

\[
I^n/I^{n+1} \cong J^n/J^{n+1} \quad \text{and} \quad B/I^n \cong C/J^n \quad \text{for} \quad n \geq 1.
\]

Furthermore the induced map

\[
\begin{array}{ccc}
B/\bigcap_{n \geq 0} I^n & \longrightarrow & C/\bigcap_{n \geq 0} J^n
\end{array}
\]

is a monomorphism.

**Proof.** By induction on $n$. The case $n=1$ is true by hypothesis. Suppose that the map $f$ induces isomorphisms $I^{n-1}/I^{n} \cong J^{n-1}/J^n$ and $B/I^{n} \cong C/J^n$. Using lemma 1.1 we obtain a commutative diagram of exact sequences