DOMINATION THEOREMS IN NON-COMMUTATIVE C*-ALGEBRAS

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Let $A$ be a Banach algebra with unit $e$. If $u, v_1, \ldots, v_n$ are elements of $A$ then $u$ is said to be dominated by $v_1, \ldots, v_n$ if there is $\gamma > 0$ such that for every $x \in A$ we have $\|ux\| \leq \gamma (\|v_1x\| + \ldots + \|v_nx\|)$. It is shown that for every $x \in A$ we have $\|ux\| \leq \gamma (\|v_1x\| + \ldots + \|v_nx\|)$. It is shown for finite-dimensional algebras that the above condition is sufficient for $u$ to belong to the left ideal generated by $v_1, \ldots, v_n$ in some superalgebra of $A$. A similar result is proved for subalgebras of the algebra of all bounded operators on a Hilbert space.

1. Introduction.

Let $A$ be a commutative Banach algebra with unit $e$. If $u, v_1, \ldots, v_n$ are elements of $A$ then $u$ is said to be dominated by $v_1, \ldots, v_n$ if there is a positive constant $\gamma$ such that for every $x \in A$ we have

\[ \|ux\| \leq \gamma (\|v_1x\| + \ldots + \|v_nx\|). \]

We call a commutative unital Banach algebra $B$ an extension of $A$ if $A$ is isometrically embedded in $B$ and the unit of $B$ is that of $A$. It is evident that (1) is a necessary condition for the existence of an extension $B$ of $A$ such that

\[ u = b_1v_1 + \ldots + b_nv_n \]
for some $b_1, \ldots, b_n \in B$. The problem whether the converse is true was known for long as the domination conjecture. Arens [1] gave the positive answer for $n = 1$. It was later Zelazko [6] who proved it for function algebras, where the algebra of all bounded functions on the Silov boundary of $A$ appeared to serve as the desired extension of $A$, common to all systems $u, v_1, \ldots, v_n$ satisfying (1). In 1982 Müller [4] provided an example showing that even for $n = 2$ the domination conjecture is false.

In this note we study the non-commutative counterpart of the problem. It should be noted that dropping the commutativity is not simply a generalization: we relax the assumptions on $A$ but at the time we naturally have to admit non-commutative extensions of $A$. To our knowledge, the problem whether (1) implies (2) is in general open even for $n = 1$ and $u = e$. We show in Section 2 that the domination conjecture is true for finite-dimensional algebras. This appears as a purely algebraic result and (1) is replaced by the apparently weaker condition

$$ux = 0 \text{ whenever } v_1x = \ldots v_nx = 0.$$ 

We wish to mention that the commutative domination conjecture is false even in finite-dimensional case (see [4]).

In Section 3 we employ certain well-known ideas of Hilbert space operator theory to show the domination conjecture in the case when $A$ is a (not necessarily self-adjoint) subalgebra of the $C^*$-algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space $H$. The result is stated under the additional assumption that $A$ contains all finite-dimensional operators.

### 2. Domination theorem for finite-dimensional algebras.

Given a vector space $X$ we write $L(X)$ for the algebra of all linear operators from $X$ into $X$.

**Lemma 1.** Let $X$ be a finite-dimensional vector space. Suppose